DOCUMENT

ESA pointing error engineering handbook

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Introduction

The ECSS Control Performance standard E-ST-60-10C, ECSS-E-ST-60-10C, provides solid and exact elements to build up a performance error budget. However, the elements recommended are not embedded in an engineering framework and thus an intermediate document developing a Pointing Error Engineering (PEE) methodology is herewith formulated, as foreseen in Note 3 of ECSS-E-ST-60-10C:

"For their own specific purpose, each entity (ESA, national agencies, primes) can further elaborate internal documents, deriving appropriate guidelines and summation rules based on the top level clauses gathered in this E-ST-60-10C standard."

The purpose of this handbook is to be used by ESA projects as reference document providing clauses, guidelines, recommendations and examples, consistent with and elaborating the E-ST-60-10C for the specific case of satellite pointing errors.

This handbook provides

- guidelines for characterization of pointing error sources,
- guidelines for analysing pointing error source contribution to the actual pointing error index,
- summation and compilation guidelines for the system pointing error budget.

Specific and quantitative performance pointing requirements are expressed as usual in the ESA Mission Requirement Document and System Requirement Document, and further broken down and engineered by the prime contractor in the various project phases.

1 Scope

This document focuses on the formulation of a consistent methodology for performing pointing error engineering on system and subsystem (SS) level in line with the definitions in ECSS-E-ST-60-10C, thus enabling systematic requirements engineering and system design as illustrated in Figure 1-1.

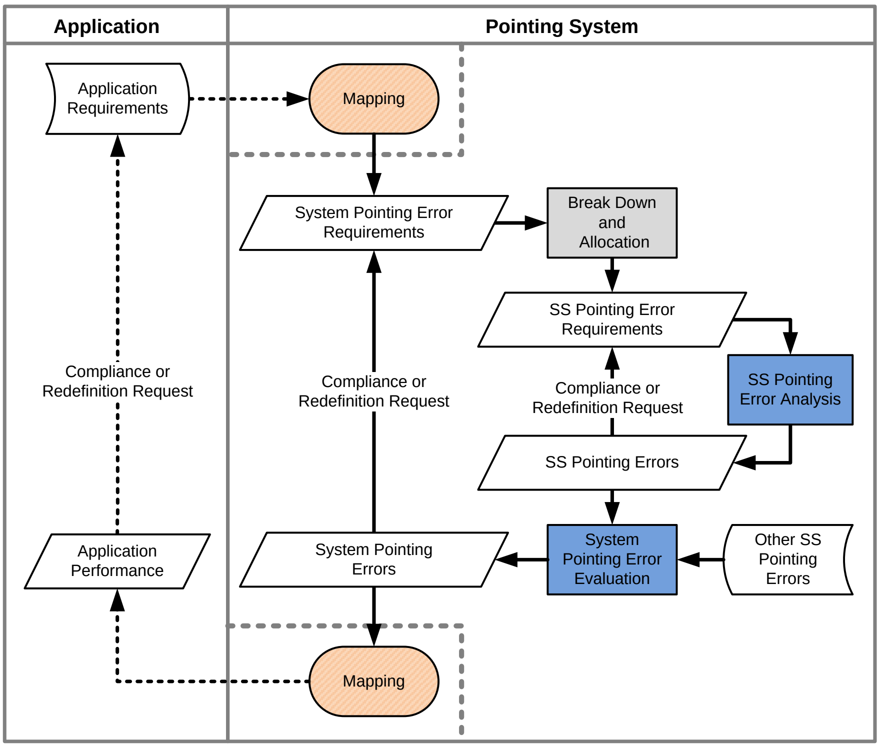

Figure 1-1: Scope of document

In this document guidelines for pointing error engineering are elaborated in terms of:

- interface definition for mapping application requirements into unambiguously formulated system pointing error requirements and vice versa,
- guidelines for characterization of pointing error sources,
- guidelines for analysing pointing error source contribution to the pointing error index of interest,
- summation and compilation of the system pointing error budget.

NOTE The actual mapping of application requirements into system pointing error requirements by means of pointing error indices defined in ECSS-E-ST-60-10C, is not treated in this document because the mapping is application specific.

2 References

2.1 ECSS standards

The following documents are called by this handbook:

ECSS-S-ST-00-01C	ECSS - Glossary of terms
ECSS-E-ST-10-09C	Space engineering - Reference coordinate system
ECSS-E-ST-60-10C	Space engineering - Control performance
ECSS-E-ST-60-20C Rev.1	Space engineering - Stars sensors terminology and performance specification

2.2 Other references

[RD-01] Lucke R.L., Sirlin S.W., San Martin A.M., "New Definition of Pointing Stability: AC and DC Effects", *The Journal of the Astronautical Sciences*, Vol. 40, No. 4, p. 557-576, 1992.

[RD-02] Pittelkau M.E., Pointing Error Definitions, Metrics, and Algorithms, *American Astronautical Society*, AAS 03-559, p. 901, 2003.

[RD-03] Bendat, J.S. und Piersol, Random Data-Analysis and Measurement Procedures, *Chichester: John Wiley & Sons,* 3rd edition, 2000.

[RD-04] ECSS-E-HB-60-10, Control Performance ECSS, "Control Performance Guidelines ECSS-E-HB-60-10", *ESA-ESTEC Requirements & Standards Division*, 2011.

[RD-05] Boyd, S. P., & Barratt, C. H., "Linear controller design: Limits of performance", *Prentice-Hall*, 1991.

[RD-06] Ott T., Fichter W., Bennani S., Winkler S., "Coherent Precision Pointing Control Design based on H_∞-Closed Loop Shaping", *8th International ESA Conference on Guidance, Navigation & Control Systems*, Karlovy Vary CZ, June 2011.

[RD-07] Bayard D. S., State-Space Approach to Computing Spacecraft Pointing Jitter, *Journal of Guidance, Control, and Dynamics,* Vol.27, No. 3, May-June 2004.

[RD-08] VEGA Space Systems Engineering, "ESA Pointing Error Handbook", *ESA Contract No.7760/88/NL/MAC*, 1993.

[RD-09] Doyle J., Francis D., Tannenbaum A., Feedback Control Theory, *Macmillan*, New York, 1992.

[RD-010] Allan D. et al., Standard Technology For Fundamental Frequency and Time Metrology, *42nd Annual Frequency Control Symposium,* 1988.

3 Terms, definitions and abbreviated terms

3.1 Terms from other documents

For the purpose of this document, the terms and definitions from ECSS-S-ST-00-01C apply.

3.2 Abbreviated terms

For the purpose of this document, the following abbreviated terms apply:

Abbreviation	**Meaning**
ABS	absolute
AD	applicable document
AKE	absolute knowledge error
AOCS	attitude and orbit control system
APE	absolute performance error
AST	analysis step
ECSS	European Cooperation for Space Standardization
ESA	European Space Agency
FOV	field of view
HW	hardware
KDE	knowledge drift error
KRE	knowledge reproducibility error
LEOP	launch and early operations phase
LOS	line of sight
LSD	linear spectral density
MKE	mean knowledge error
MPE	mean performance error
MRD	mission requirements document
NA	not applicable
PEC	pointing error contributor
PES	pointing error source
PDE	performance drift error

PDF	probability density function
PRE	performance reproducibility error
PSD	power spectral density
RD	reference document
RKE	relative knowledge error
RMS	root mean square
RPE	relative performance error
SS	subsystem
STA	stability
STR	star tracker
SRD	system requirements document
SW	software
WM	windowed mean
WMS	windowed mean stability
WV	windowed variance

3.3 Symbols

The following symbols are used in this handbook:

Symbol	Meaning
$\Vert \dots \Vert$	norm
$\langle \dots \rangle$	time average
Δt	window time
Δt_D	drift reset time interval
Δt_s	stability time
$\delta(\dots)$	Dirac-delta function
ε_{index}	zero mean pointing error per index (APE, RPE, …)
ε_D	drift error
$\Im\{\dots\}$	Fourier transform
$\mathcal{L}\{\dots\}$	Laplace transform
σ	standard deviation
σ_{BC} , σ_{CC}	standard deviation of time-constant PEC described as random variable
σ_{CRP}	standard deviation of time-random PEC described as random process
σ_{CR}	standard deviation of time-random PEC described as random variable
σ_{SRP}	standard deviation of time-random PES described as random process

σ_{SC}	standard deviation of time-constant PES described as random variable
σ_{SR}	standard deviation of time-random PES described as random variable
σ^2	variance
μ	mean value
μ_{BC} , μ_{CC}	mean value of time-constant PEC described as random variable
μ_{CRP}	mean value of time-random PEC described as random process
μ_{CR}	mean value of time-random PEC described as random variable
μ_{SRP}	mean value of time-random PES described as random process
μ_{SC}	mean value of time-constant PES described as random variable
μ_{SR}	mean value of time-random PES described as random variable
Ψ^2	mean square value
A	amplitude
B	bias
$BM(\ldots)$	bimodal probability density function
C	boundary on uniform distribution
C_{ee}	covariance function
D	drift rate
$E[\ldots]$	expected value of []
$e(t)$	pointing error depending on time t
e_{index}	pointing error per index
e_c	time-constant PEC
$e_c(t)$	time-random PEC
$e_K(t)$	pointing knowledge error
$e_k = e(k)$	pointing error depending on the ensemble realization index k
$e(k,t)$	pointing error depending on the ensemble realization k and time t
$e_k(t)$	pointing error realization with index k
$\{e_k(t)\}$	ensemble of pointing error realizations $e_k(t)$
$e_P(t)$	pointing performance error
e_r	pointing error requirement
$e_{r,index}$	pointing error requirement per index
e_s	time-constant PES
$e_s(t)$	time-random PES
$F_{metric}(\omega,\ldots)$	weighting function for time-windowed signal metric (ABS, WME, …)
$\tilde{F}_{metric}(s,\ldots)$	rational approximation of weighting function

f	frequency in [Hz]
f_N	Nyquist frequency in [Hz]
$G(\mu,\sigma)$	Gaussian probability density function
$G_{ee}(f)$	single-sided power spectral density in [unit²/Hz]
$G_{ee}(\omega)$	single-sided power spectral density in [unit²/rad s⁻¹]
$H(j\omega)$	linear time-invariant transfer function
$h(t)$	impulse response
k	Index of specific ensemble realization
$max[\ldots]$	maximum value of […]
$min[\ldots]$	minimum value of […]
$p(\ldots)$	probability density function
$p(\ldots\mid\ldots)$	probability density function depending on some event
$p_k(\ldots)$	conditional probability density function depending on realization index k
p_{BC} , p_{CC}	probability density function of time-constant PEC
p_{CR}	probability density function of time-random PEC
p_{SC}	probability density function of time-constant PES
p_{SR}	probability density function of time-random PES
$P(\ldots)$	probability distribution function
P_{ee}	linear spectral density
P_c	level of confidence
$U(\ldots, \ldots)$	uniform probability density function
$R(\ldots ,\ldots)$	Rayleigh probability density function
R_{ee}	autocorrelation function
S_{ee}	double-sided power spectral density
ω	frequency in [rad/s]

4 Pointing error: from sources to system performance

4.1 Pointing error sources and contributors

A pointing error can be considered as response of a system to external or internal physical phenomena affecting the system pointing performance as illustrated in Figure 4-1.

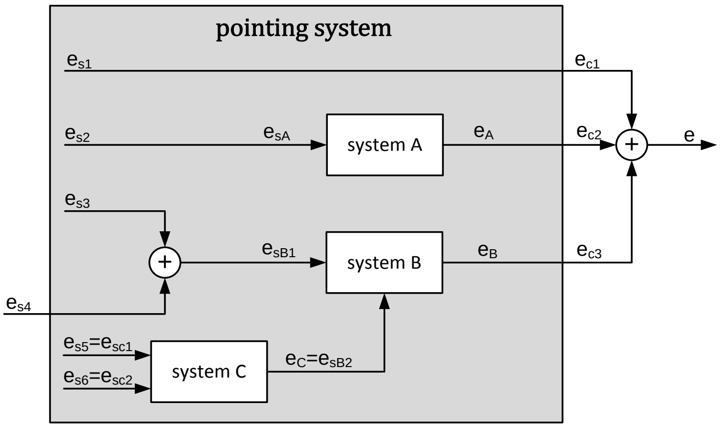

Figure 4-1: Pointing error source transfer

Physical phenomena affecting pointing performance and thus the pointing error e are referred to as ***pointing error source*** (PES) and are denoted as e_s. A PES is either constant in time (time-constant), random in time (time-random) and/or random in its realization (ensemble-random).

A ***pointing error contributor*** (PEC), denoted as e_c, represents the actual contribution of one or more pointing error sources e_s to the overall pointing error e.

A PES becomes a PEC after pointing system transfer, through e.g. the following transformations:

- coordinate frame,
- control system,
- structural.

In order to analyse pointing performance, a pointing system is broken down into subsystems with individually controlled (active or passive) transfer properties (see Figure 4-1). The pointing error e is the sum of the different PEC.

A selection of exemplary time-random PES are listed in Table 4-1 and a selection of time-constant pointing error sources are listed in Table 4-2. An example for satellite PES transfer is given in Annex B.

NOTE In terms of star sensors, ECSS-E-ST-60-20C Rev1 gives an overview of relevant PES and provides guidelines for their characterization.

Table 4-1: Time-random PES

time-random PES – $e_s(t)$
Environmental disturbances (e.g. solar pressure noise)
Payload intrinsic error sources (e.g. optical filter wheel, cryogenic cooler)
Drive mechanisms (e.g. solar array, instrument, antenna, filter wheel)
Actuator intrinsic error sources (e.g. reaction wheel imbalances, thrusters noise)
Sensor bias and noise (e.g. star tracker, gyro, accelerometer, metrology, GPS)
Structure thermo-mechanical deformations (e.g. due to orbiting)
System dynamics induced errors (e.g. sloshing, flexible modes)

Table 4-2: Time-constant PES

time-constant PES - e_s
Misalignments (e.g. payload, sensor, etc.)
Calibration uncertainty (e.g. sensor bias)

4.2 Time-windowed pointing errors

It is referred to the absolute pointing error when specifying or analysing pointing error requirements for every point in time throughout the lifetime of a system. In practice and as stated in ECSS-E-ST-60-10C also pointing errors over defined time windows are important and the relation of those windowed errors with respect to each other. That results in the necessity of characterizing time-dependent pointing errors as illustrated in Figure 4-2.

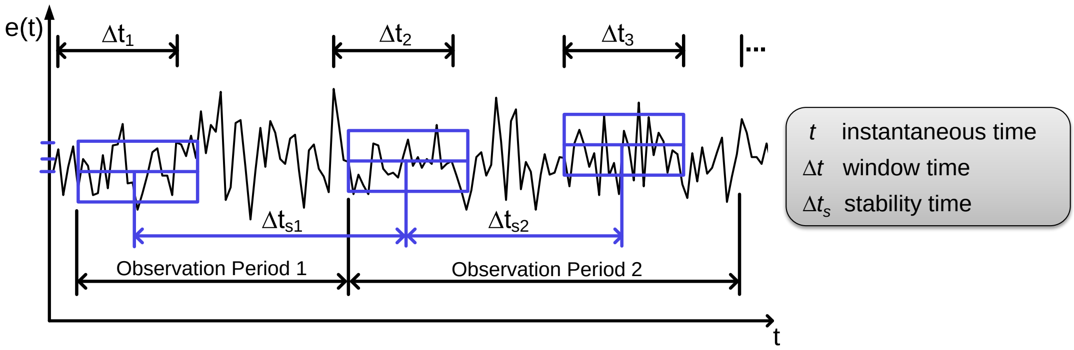

Figure 4-2: Time dependency of pointing errors

It can be distinguished between three time-dependencies of pointing errors:

- **Instantaneous time *t*:** pointing error at any point in time *t* during system lifetime or a defined observation period.
- **Window time Δ*t*:** pointing error within a time window Δ*t*, whereas the time window can occur at any point in time *t* during system lifetime or a defined observation period.
- **Stability time Δt_s:** pointing error describing stability, thus the relative error, among pointing errors in time-windows of length Δ*t*. The time-windows are separated by a time difference of length Δt_s, and can occur at any point in time t during system lifetime or a defined observation period.

Table 4-3: Definition of pointing error indices

Index	Name	Definition
AKE	Absolute Knowledge Error	Difference between the actual parameter (attitude, geolocation, etc.) and the known (measured or estimated) parameter in a specified reference frame.
APE	Absolute Performance Error	Difference between the target (commanded) parameter (attitude, geolocation, etc.) and the actual parameter in a specified reference frame.
MKE	Mean Knowledge Error	Mean value of the AKE over a specified time interval Δt.
MPE	Mean Performance Error	Mean value of the APE over a specified time interval Δt.
RKE	Relative Knowledge Error	Difference between the APE at a given time within a time interval, Δt, and the MKE over the same time interval.
RPE	Relative Performance Error	Difference between the APE at a given time within a time interval, Δt, and the MPE over the same time interval.
KDE	Knowledge Drift Error	Difference between MKEs taken over two time intervals separated by a specified time, Δt_s, within a single observation period.
PDE	Performance Drift Error	Difference between MPEs taken over two time intervals separated by a specified time, Δt_s, within a single observation period.
KRE	Knowledge Reproducibility Error	Difference between MKEs taken over two time intervals separated by a specified time, Δt_s, within different observation periods.
PRE	Performance Reproducibility Error	Difference between MPEs taken over two time intervals separated by a specified time, Δt_s, within different observation periods.

The time-dependent pointing errors are defined in ECSS-E-ST-60-10C and summarized in Table 4-3. Note that a KDE and KRE pointing error index is added in this handbook for being complete.

The comprehensive set of pointing error indices, categorized in knowledge or performance errors and depending on instantaneous, window and stability time, is formulated in Table 4-4.

NOTE The necessity of analysing time-dependent pointing errors has its origin in actual instrument and observation requirements of pointing systems, e.g. a satellite and its payload, as discussed in [RD-01] and [RD-02].

Table 4-4: Mathematical formulation of pointing error indices

Pointing Error Indices	
index	***instantaneous***
$e_{APE}(t)$	$= e_P(t)$
$e_{AKE}(t)$	$= e_K(t)$
$e_{MPE}(t, \Delta t)$	$= \overline{e_P}(t, \Delta t)$
$e_{MKE}(t, \Delta t)$	$= \overline{e_K}(t, \Delta t)$
$e_{RPE}(t, \Delta t)$	$= e_P(t) - \overline{e_P}(t, \Delta t)$
$e_{RKE}(t, \Delta t)$	$= e_K(t) - \overline{e_K}(t, \Delta t)$
$e_{PDE}(t, \Delta t_1, \Delta t_2, \Delta t_s)$ $e_{PRE}(t, \Delta t_1, \Delta t_2, \Delta t_s)$	$= \overline{e_P}(t, \Delta t_1) - \overline{e_P}(t + \Delta t_s, \Delta t_2)$
$e_{KDE}(t, \Delta t_1, \Delta t_2, \Delta t_s)$ $e_{KRE}(t, \Delta t_1, \Delta t_2, \Delta t_s)$	$= \overline{e_K}(t, \Delta t_1) - \overline{e_K}(t + \Delta t_s, \Delta t_2)$

Δt_s	*stability time*	e_{index}	*instantaneous error*
Δt_s	*stability time*	$e_K(t)$	*knowledge error signal*
		$e_P(t)$	*performance error signal*

time average: $$\bar{e}(t, \Delta t) = \langle e(t) \rangle_{\Delta t} = \frac{1}{\Delta t} \int_{t-\Delta t/2}^{t+\Delta t/2} e(t)\, dt$$

instantaneous time

window time

stability time

5 Pointing error engineering framework

5.1 Overview

Pointing error engineering covers the engineering process of establishing system pointing error requirements, their systematic analysis throughout the design process, and eventually compliance verification. In terms of specification, analysis and verification, it is necessary to be aware of the whole pointing error engineering cycle. That is, for specification of pointing error requirements relevant analysis and verification methods need to be identified and vice versa.

5.2 Methodology

The flow diagram in Figure 5-1 gives a schematic overview. The process starts with mapping Application Requirements, as specified by the user, into System Pointing Error Requirements. The System Pointing Error Requirements preferably follow the classification provided in Table 4-4. The compliance of the system pointing error requirements is analysed by estimating and combining the different occurring error sources in the analysis steps (AST) 1 to 4.

NOTE The mapping process is not further treated in this handbook because it is application specific. However, in the future it is intended to provide a document with exemplary requirement mapping cases for different types of satellite missions. Apart from that, this handbook covers the whole pointing error engineering cycle providing a framework with mathematical elements, engineering methods and conventions.

As discussed in section 4 each subsystem (SS) within the pointing system is analysed in terms of their Pointing Error Source (PES) transfer characteristics to compile a pointing error budget. Hence, the first analysis step, ***AST-1***, in pointing error analysis is to identify and characterize the PES.

In the second analysis step, ***AST-2***, it is analysed how the different PES contribute to the pointing error, as already mentioned in section 4. These Pointing Error Contributors (PEC) are obtained by a transformation, which depends on the system under evaluation. The transformation can for example be a change in reference frame. The transfer can also be a dynamic process, such as a satellite AOCS. The transfer characteristics of each system are tuneable to a certain extent and thus can be used to perform trade-offs with the aim of making pointing errors compliant with their requirement. AST-2 is elaborated in section 9.

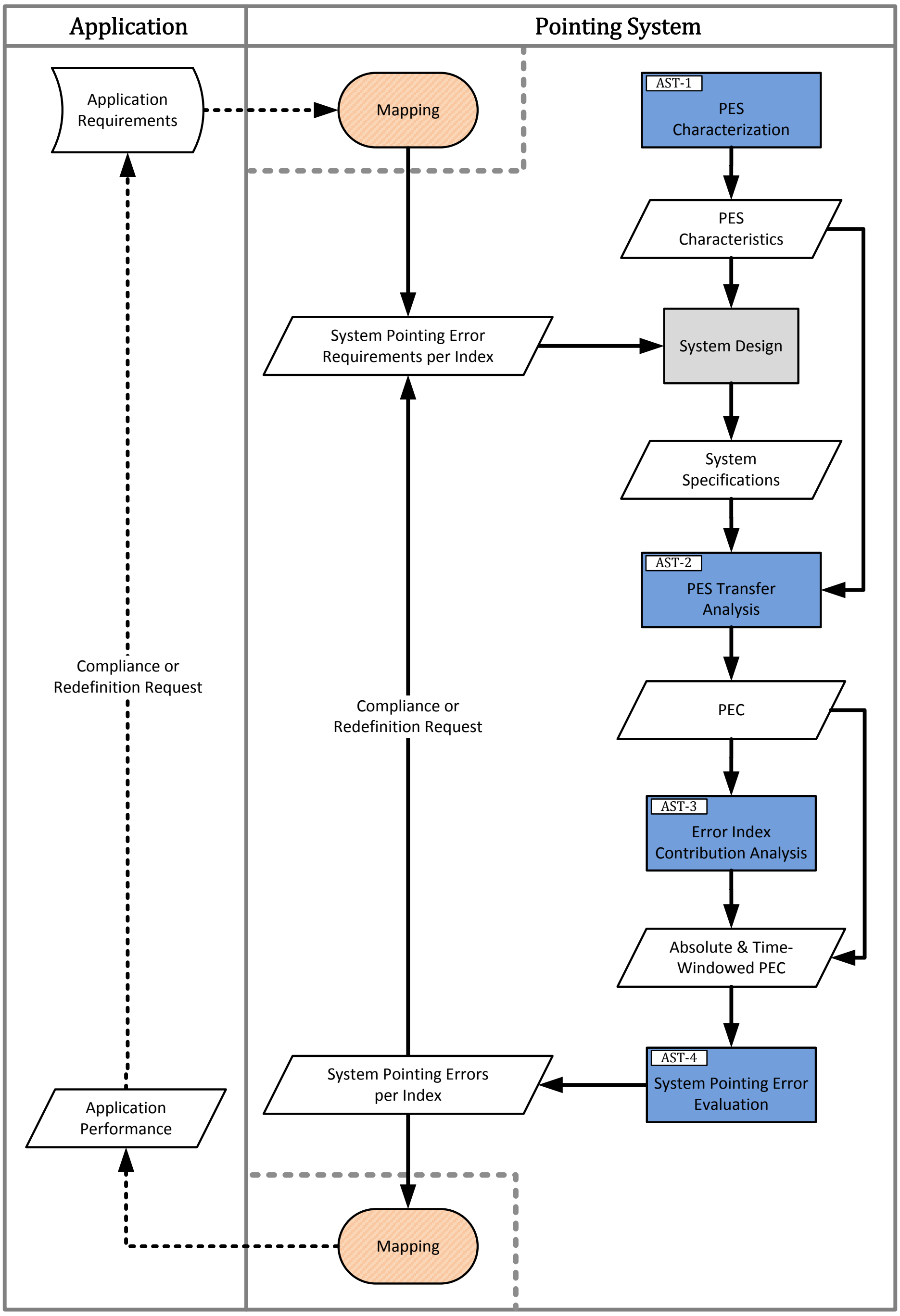

Figure 5-1: Pointing error engineering methodology structure

In early mission phases some of the transfer characteristics might not be available thus step AST-2 can be discarded and the system needs to be considered to transfer the PES one-to-one. That means PES migrate directly to PEC. On the other hand, in late development phases detailed knowledge of complex transfer characteristics might be available for time domain and frequency domain analysis. In the case of frequency domain analysis, simplified linearized models of the transformation process may be derived, as introduced in section 9.2.

The third step, ***AST-3***, determines the contribution of the PEC to the pointing error indices. It is elaborated in section 10.

AST-3 can be skipped for random time-constant PES because it does not depend on time. Moreover, AST-3 can also be skipped for the analysis of the Absolute Pointing Error because it only depends on the instantaneous time, and not on windowed or windowed stability time.

The fourth step, ***AST-4***, compiles the absolute and different time-windowed pointing error contributors to obtain an estimate of the overall pointing error, which is then compared with the requirement. The analysis step is elaborated in section 11.

In complex cases, the pointing system can be broken down in several subsystems. The Analysis Steps (AST) from Figure 5-1 can then be applied to each subsystem as shown in Figure 5-2 for the example of an AOCS subsystem. The steps AST-1 to AST-4 are applied on each subsystem of the pointing system. At the end AST-4 is performed again on pointing system level in order to compile and evaluate the overall pointing error budget.

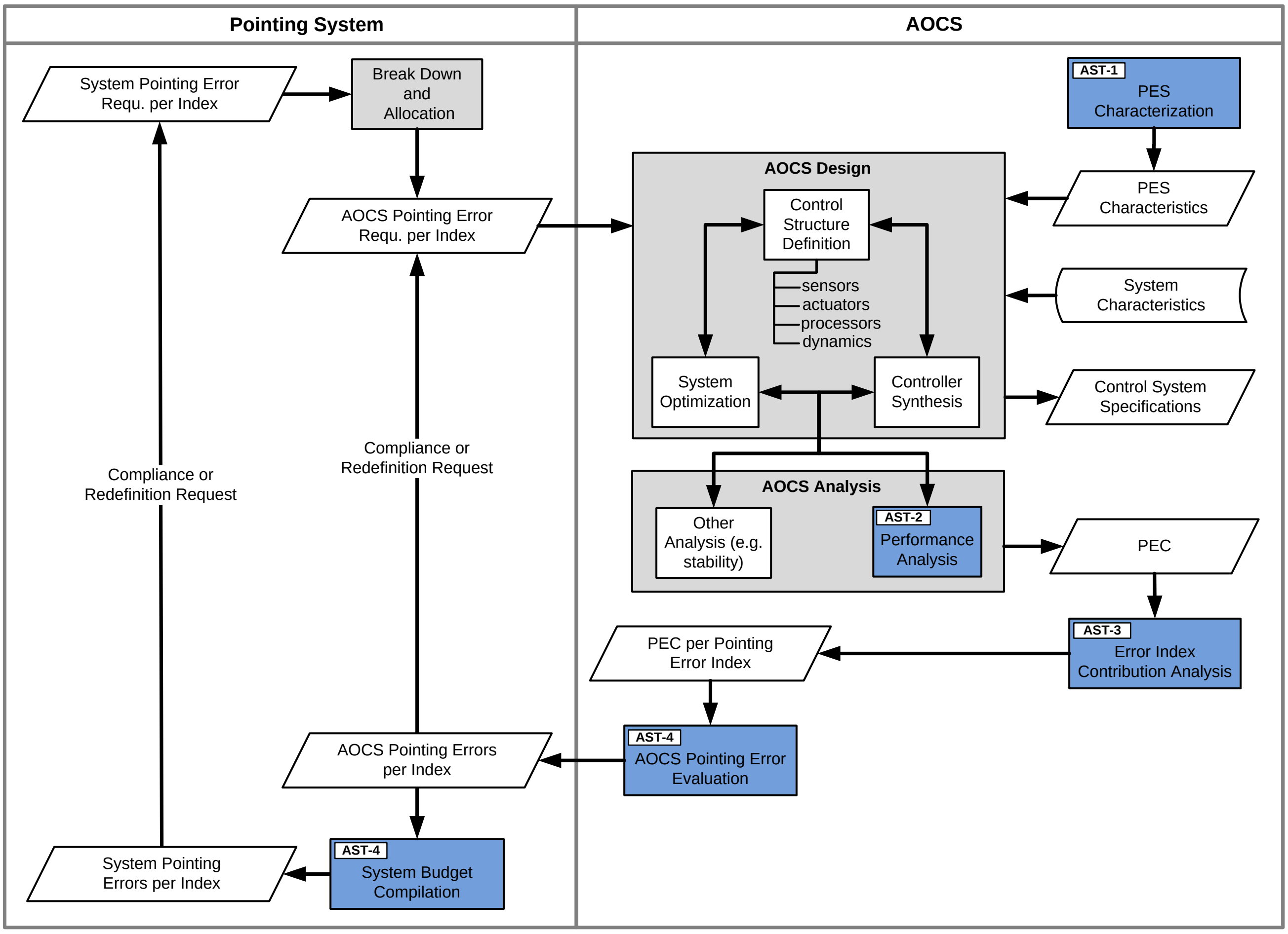

Figure 5-2: Pointing error engineering in AOCS

In this handbook a framework to characterize PES and analyse their transfer behaviour is defined. Thereafter, AST-1 to AST-4 are elaborated and guidelines are introduced.

5.3 Framework elements

5.3.1 Overview

The framework elements are the "design language" for pointing error engineering. It is a consistent mathematical framework to describe relevant properties of PES for analysing their system transfer and eventually to quantify the overall pointing error indices. The framework consists of methods used in probability theory to describe properties of random physical phenomena acting as PES.

5.3.2 Mathematical elements

5.3.2.1 Overview

Mathematical elements necessary to perform pointing error engineering are summarized and introduced in this section 5.3.2. This includes random variables, probability functions and random process theory. A comprehensive discussion of the topics is given in [RD-03].

5.3.2.2 Random variable

A pointing error has random unpredictable magnitudes, with all possible magnitude values making up the sample space. The magnitude values can either vary randomly in time (time-random) or in the ensemble of realizations (ensemble-random). The random error magnitudes thus represent a random variable taking on real numbers between -∞ and ∞ associated to each error sample point in the sample space. The random variable can either be:

- $e(k)$ thus depending on the realization k,
- $e(t)$ thus depending on the point in time t,
- $e(k,t)$ thus depending on both, the realization k and point in time t.

NOTE The notation $e(k)$ is used to introduce random variables and probability. It represents the case where e varies due to any random-property making up a sample space with k realizations. If the random property is linked to time, then this is explicitly denoted by replacing k with t. In the more complex case is the dependency of the pointing error depends on both k and t, thus $e(k,t)$.

5.3.2.3 Probability distribution and density function with statistical properties

5.3.2.3.1 Overview

In order to perform pointing error analysis the error sample space needs to be characterized. In that respect probability functions are assigned to describe the error sample space and thus the random variable or process, cf. [RD-03].

5.3.2.3.2 Probability Distribution Function

In the general case, the probability distribution function describes the probability that a pointing error $e(k)$ is less than a defined required error value e_r, meaning that $e(k) < e_r$. The probability assigned to the set of points k in the sample space that satisfy the inequality are described by the probability distribution function:

$$P(e) = \text{Prob}\left[e(k) < e_r\right] \quad \textbf{(5-1)}$$

with $P(-\infty) = 0$ and $P(\infty) = 1$.

5.3.2.3.3 Probability Density Function

In terms of pointing error analysis it is more convenient to work with probability density functions (PDF). If a random variable has a continuous range of values, the PDF is defined to be the first order derivative of the probability distribution function:

$$p(e) = \frac{dP(e)}{de} \quad \textbf{(5-2)}$$

with $\int_{-\infty}^{\infty} p(e)de = 1$ and $p(e) \geq 0$.

NOTE The probability density function *p(e)* is permitted to represent a Dirac-delta function.

5.3.2.3.4 Statistical Properties

Statistical properties describe the random variables and are a function of the underlying PDF. Concerning this handbook three different properties are of interest, all defined by the expected value.

The mean value μ_e of *e(k)* is defined by:

$$\mu_e = E[e(k)] = \int_{-\infty}^{\infty} e \; p(e)de \quad \textbf{(5-3)}$$

The mean square value ψ_e^2 of *e(k)* is defined by:

$$\psi_e^2 = E[e(k)^2] = \int_{-\infty}^{\infty} e^2 p(e)de \quad \textbf{(5-4)}$$

The variance σ_e^2 of *e(k)* is defined by:

$$\sigma_e^2 = \psi_e^2 - \mu_e^2 = E[(e(k) - \mu_e)^2] = \int_{-\infty}^{\infty} (e - \mu_e)^2 p(e)de \quad \textbf{(5-5)}$$

The RMS value corresponds to:

$$e_{rms} = \sqrt{\psi_e^2} \quad \textbf{(5-6)}$$

NOTE In terms of pointing error analysis the RMS value is usually considered with zero-mean value. If this is the case, this should be clearly mentioned.

5.3.2.3.5 Summary of Statistical Properties with Respective PDF

There are various PDF for describing the sample space of a pointing error. However, it is practical to describe a pointing error in line with the most common ones. In this respect a summary of statistical properties with the respective PDF is given in Table 5-1.

Table 5-1: Statistical properties with respective PDF

PDF	p(e)	μ_e	σ_e
Discrete	$\delta(e-\mu_e)$	$\mu_\delta = \mu_e$	$\sigma_\delta = 0$
Uniform	$U(e_{\min}, e_{\max}) = (e_{\max} - e_{\min})^{-1}$, $\forall\ e_{\min} \le e \le e_{\max}\ \ otherwise\ 0$	$\mu_U = \frac{e_{\min} + e_{\max}}{2}$	$\sigma_U = \frac{e_{\max} - e_{\min}}{\sqrt{12}}$
Bimodal	$BM(A) = \left[\pi\sqrt{A^2 - e^2}\right]^{-1}$, $\forall\ \lvert e\rvert < A\ \ otherwise\ 0$	$\mu_{BM} = 0$	$\sigma_{BM} = \frac{1}{\sqrt{2}} A$
Gaussian (normal)	$G(\mu_e, \sigma_e) = \frac{1}{\sigma_e\sqrt{2\pi}} \exp\left[-\frac{(e-\mu_e)^2}{2\sigma_e^2}\right]$	$\mu_G = \mu_e$	$\sigma_G = \sigma_e$
Rayleigh	$R(e, \sigma_R) = \frac{e}{\sigma_R^2} \exp\left[-\frac{e^2}{2\sigma_R^2}\right]$, $\forall\ e \ge 0$	$\sqrt{\frac{\pi}{2}}\sigma_R$	$\sqrt{\frac{4-\pi}{2}}\sigma_R$

5.3.2.3.6 Conditional Probability

Conditional probability is used to describe errors that are random with respect to two dependent variables. Thus randomness over the ensemble realizations *k* as well as in time *t* can be described.

The topic of conditional probability is summarized in ECSS-E-ST-60-10C and discussed in detail in [RD-03] and [RD-04].
In this handbook a conditional PDF is denoted as:

$$p_k(e) = \int_{-\infty}^{\infty} p(e \mid k)\, p(k)\, de \tag{5-7}$$

with the statistical properties $\mu_e(k)$ and $\sigma_e(k)$ depending on the realization index *k*.

NOTE Instead of using conditional probability it is possible to describe a pointing error as stationary random process, cf. section 5.3.2.4. However, in order to do so, mixed statistical interpretation needs to be specified in the pointing error requirement.

5.3.2.4 Random process

A random pointing error process $\{e_k(t)\}$ is an ensemble of *k* sampling function realizations that are random in time *t* (time-random) and random in its ensemble of realizations (ensemble-random). The ensemble is the set {…} of all realizations *k* of the random pointing error $e_k(t)$. The probability properties of a random process are described by the ensemble statistical quantities (e.g. mean or variance) at fixed values of *t*, where $e_k(t)$ is a random variable over the index *k*. In general, the statistical quantities are different at different times *t*. If the statistical quantities are equal for all *t* the random process is said to be **stationary**. In this handbook, if referred to random processes, it is implicitly assumed that it is stationary.

A time-random pointing error $e_k(t)$ with ensemble-random realizations *k* can be described as a stationary random process if time series data of the respective PES is available, and the conditions stated in [RD-03], for stationary random processes description, are fulfilled.

A stationary random process is described by its PDF $p(e)$. In practice most stationary random processes have a Gaussian PDF and thus are completely defined by their mean value and covariance respectively:

$$\mu_e = E[e_k(t)] = \int_{-\infty}^{\infty} e \; p(e)de \tag{5-8}$$

$$C_{ee}(\tau) = E[(e_k(t)e_k(t+\tau)] - \mu_e^2 = R_{ee}(\tau) - \mu_e^2 \tag{5-9}$$

where the autocorrelation is defined as:

$$R_{ee}(\tau) = E[(e_k(t)e_k(t+\tau)] = \iint_{-\infty}^{\infty} e_1 e_2 \; p(e_1, e_2) \; de_1 de_2 \tag{5-10}$$

with $e_1 = e_k(t)$ and $e_2 = e_k(t+\tau)$. Note that the statistics are independent of t and that the covariance function $C_{ee}(\tau)$ represents the variance of the random process for $\tau = 0$.

A stationary random process is **ergodic** in case the ensemble probability characteristics can be determined by time averages of arbitrary realizations k. In terms of mean value and covariance this means that:

$$\mu_e(k) = \mu_e \qquad \text{with} \qquad \mu_e(k) = \lim_{T\to\infty} \frac{1}{T} \int_0^T e_k(t)dt \tag{5-11}$$

$$C_{ee}(\tau,k) = C_{ee}(\tau) \qquad \text{with} \qquad C_{ee}(\tau,k) = R_{ee}(\tau) - \mu_e^2(k)$$
$$R_{ee}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T e_k(t)e_k(t+\tau)dt \tag{5-12}$$

If a stationary random pointing error process is ergodic, pointing error analysis can be simplified because the probability characteristics can be determined based on one time series instead of an ensemble of time series.

NOTE A random process is stationary if its PDF is not a function of time. This is usually the case for time-invariant operational conditions. If the conditions change throughout the lifetime of the pointing system, a quasi-stationary process needs to be identified. Quasi stationary process can be determined via worst case behaviour of the PES over a specified period of interest or its statistical properties are described as random variables themselves. An alternative approach is to require error source signals to have stationary behaviour, e.g. by controlling operational conditions of the PES. For time-random errors that have transient behaviour a non-stationary random process description and analysis is inevitable, cf. [RD-03].

5.3.2.5 Power spectral density

The frequency domain characteristics of a random stationary process are described by means of its power spectral density (PSD). This becomes important when considering time- windowed pointing errors because a windowing in the time domain is equivalent to a low pass filtering in the frequency domain. This enables mathematically exact analysis of time dependent pointing errors as introduced in section 10.

In the frequency domain the power of a signal is equivalent to the area underneath the even double-sided power spectral density PSD function S_{ee}:

$$\psi_e^2 = R_{ee}(0) = \Im^{-1}\{S_{ee}(f)\}\Big|_{\tau=0} = \int_{-\infty}^{\infty} S_{ee}(f)df = \int_0^{\infty} 2S_{ee}(f)df = \int_0^{\infty} G_{ee}(f)df \tag{5-13}$$

where the mean square value corresponds to the autocorrelation function at its maximum, which occurs for $\tau=0$.

There are different notations for a PSD as can be seen in Eq. (5-13). The double-sided PSD is denoted as S_{ee}, the single-sided PSD as G_{ee}, but both are in [unit²/Hz]. In some literature the square-root of the single-sided PSD $P_{ee}=\sqrt{G_{ee}}$ in [unit/√Hz)], also called Linear Spectral Density (LSD), is used. In this handbook it is mainly referred to the single-sided PSD, G_{ee}.

NOTE If a zero-mean random stationary process is ergodic, its PDF can be characterized only by the knowledge of its PSD because a zero-mean Gaussian PDF only depends on σ_e.

5.3.3 Statistical interpretation in context of framework

5.3.3.1 Overview

The properties of physical phenomena, and thus the pointing errors and their sources, are described in terms of their probability characteristics. To make clear which property and corresponding probability characteristic is described, it is necessary to choose the statistical population by specifying one of the three statistical interpretations introduced in ECSS-E-ST-60-10C:

- mixed
- ensemble
- temporal

Each statistical interpretation requires the description of a different probability characteristic, representing a different property of the pointing error and thus a different statistical population. Hence the statistical interpretation needs to be specified in the requirement formulation such that analysis is performed in line with it. In the following the statistical interpretations, which are defined in ECSS-E-ST-60-10C, are summarized and put into context with the random variable description in section 5.3.2.2, by $e(k,t)$, and the random process description in section 5.3.2.4, by $\{e_k(t)\}$.

A PES has probability characteristics owing to its randomness in time (time-random), randomness in its ensemble of realizations (ensemble-random) or both. In a pointing system several PES are usually acting with different probability characteristics, but pointing error requirements are defined in line with one statistical interpretation. Hence, if a PES is time-random and ensemble-random worst case assumptions are necessary for one or the other property to guarantee evaluation in line with the specified statistical interpretation of the pointing error requirement. This is clarified in sections 5.3.3.2 to 5.3.3.4.

NOTE At the respective pointing error analysis steps shown in Figure 5-1, it is important to express the PES, PEC and pointing error indices in terms of the required statistical interpretation.

5.3.3.2 Mixed interpretation

In the mixed interpretation one considers the probability P greater or equal to a level of confidence P_c such that the ensemble of pointing error realizations $\{e_k(t)\}$ or $e(k,t)$ is less than a required error value e_r in its ensemble of realizations k and in time t:

$$\mathrm{Prob}\left[\ \left|\{e_k(t)\}\right| < e_r\ \right] \geq P_c \quad \text{or} \quad \mathrm{Prob}\left[\ \left|e(k,t)\right| < e_r\ \right] \geq P_c \tag{5-14}$$

NOTE In the mixed interpretation it is required to choose the statistical population such that the pointing error is described with respect to its ensemble-random and time-random probability characteristics concurrently. The mixed interpretation is equivalent to the ensemble or temporal interpretation if a PES does not randomly vary in time (time-random) or over its ensemble of realizations (ensemble-random) respectively.

5.3.3.3 Ensemble interpretation

In the ensemble interpretation the probability P greater or equal to a level of confidence P_c is considered, such that a realization k of the ensemble of pointing error realizations $\{e_k(t)\}$ or $e(k,t)$ is less than a required error value e_r for all times t:

$$\mathrm{Prob}\left[\ \left|e_{\max}(k)\right| < e_r\ \right] \geq P_c \quad \text{with} \quad e_{\max}(k) = \max_t\left[\{e_k(t)\}\right] \quad \text{or} \quad e_{\max}(k) = \max_t\left[e(k,t)\right] \tag{5-15}$$

NOTE In the ensemble interpretation it is not required to choose the statistical population thus that the time-random properties of the pointing error are described, but rather the maximum value in time of each realization k as stated in Eq.(5-15). This corresponds to a description of the ensemble-random probability characteristics.
The maximum value usually occurs at different points in time for each realization of a PES with time-random properties. Hence 'ensemble' refers to the group of maximum values of the pointing error realization $e_k(t)$ and not to the magnitude distribution of a time-random error realization.
If a PES is time-constant, $\{e_k(t)\}=e(k)$ or $e(k,t)=e(k)$, and then the maximum value of one realization is constant.

5.3.3.4 Temporal interpretation

In the temporal interpretation the probability P greater or equal to a level of confidence P_c is considered, such that the entire ensemble of pointing error realizations $\{e_k(t)\}$ or $e(k,t)$, or just the worst case realization, with realization index k, is less than a required error value e_r for a fraction of time t:

$$\mathrm{Prob}\left[\ \left|e_{\max}(t)\right| < e_r\ \right] \geq P_c \quad \text{with} \quad e_{\max}(t) = \max_k\left[\{e_k(t)\}\right] \quad \text{or} \quad e_{\max}(t) = \max_k\left[e(k,t)\right] \tag{5-16}$$

NOTE In the temporal interpretation it is not required to choose the statistical population thus that the ensemble-random probability characteristics of a pointing error are described, but rather the time-random. Hence, if the realization of a pointing error is random, with the ensemble index k, the time-random properties of its worst case realization k need to be described as stated in Eq.(5-16).

6 Pointing error requirement formulation

6.1 Overview

A pointing error requirement is the specification of probability that the system output of interest does not deviate by more than a given amount from the target output with a level of confidence P_c. In this context a pointing error can refer to the actual system deviation and thus performance, or the determination of the actual deviation thus knowledge.

This section 6 outlines the required parameters for unambiguously defining pointing error requirements. In this context the major terms of ECSS-E-ST-60-10C are recalled for clarification.

6.2 Specification parameters

The following parameters and formulations need to be specified to unambiguously define pointing error requirements:

- required error value e_r;
- statistical interpretation: ensemble, temporal, mixed;
- angular deviation per axis or half-cone angle deviation in respective pointing reference frame on which the requirement is imposed;
- error index to be constrained, cf. section 4.2:
  - pointing performance error: APE, MPE, RPE, PDE, PRE, or
  - pointing knowledge error: AKE, MKE, RKE, KDE, KRE;
- window time Δt and/or stability time Δt_s except in case of APE and AKE;
- evaluation period;
- the level of confidence P_c thus that $\mathrm{Prob}\left[\ |e_{index}| < e_{r,index}\right] \ \geq P_c$;
- optional: PSD requirement $G_R(f)$ within respective bandwidth, with the restriction of f > 0.

  NOTE Pointing requirements specified as PSD function $G_R(f)$ are usually self-contained, meaning that there is no need to specify other parameters.

6.3 Notes on requirement specification parameters and formulations

6.3.1 Reference frame and axis

The pointing scene of a satellite is illustrated in Annex A. The pointing requirements are specified in terms of half-cone angle deviations, in the following called Line-Of-Sight (LOS) error, or as rotational angle deviations per axes of the pointing reference frame, cf. ECSS-E-ST-10-09C, in the following called axis error. In this respect small angles are assumed thus that the attitude error has vector properties.

It is important to take special care when defining a pointing requirement per axes or LOS. This is illustrated in this section. Consider a pointing error made up of individual errors per axis of the pointing system reference frame:

$$e_x(t) = A\cos(\omega_0 t + \varphi)\,, \tag{6-1}$$

$$e_y(t) = A\sin(\omega_0 t + \varphi)\,, \tag{6-2}$$

$$e_z(t) = 0\,. \tag{6-3}$$

If the total error e is taken as the quadratic sum of the individual errors, then the LOS error becomes:

$$e = \sqrt{e_x^2(t) + e_y^2(t)} = A\sqrt{\cos^2(\omega_o t + \varphi) + \sin^2(\omega_o t + \varphi)} = A\,, \tag{6-4}$$

which is typically given e.g. in case of a dominating nutation motion.

The LOS pointing error in this case only consists of a time-constant mean value and thus has no frequency content for arbitrarily large A. This means that the PSD of the error is:

$$G_{ee}(\omega) = \mu_e^2 \delta(\omega)\,, \tag{6-5}$$

although the pointing errors per axis would have a PSD according to Eq.(8-3), at the frequency ω_0.

This means that in case of LOS pointing error specification, the analysis in terms of pointing error index contribution, as introduced in section 10, would result in no contributions to indices being affected by frequency content other than at ω_0=0 rad/s, i.e. RPE, RKE, KDE, PDE, KRE, PRE.

On the other hand pointing error specification per axes would result in an analysis regarding the PSD as it is specified in Eq.(8-3).

Thus, e.g. considering the situation of a “nutating satellite”, care needs to be taken when specifying PSD requirements on the LOS. The PSD of the LOS might be compliant with the requirement by considering the error magnitude e only, but considering the pointing scene in Annex A the pointing error vector $\mathbf{e_1}$ rotates on the target plane. If an instrument does PSD measurements along the LOS, it might encounter a disturbance PSD due to the rotating pointing error vector, although the vector magnitude, and thus the LOS pointing error e, is constant. This is e.g. the case if a PSD is computed from payload intensity measurements of a CCD, which usually has non-uniform pixel sensitivity.

NOTE The guidelines in ECSS-E-ST-60-10C, section 4.2.4, need to be considered when mapping a half-cone pointing error requirement (of e.g. a payload boresight) into rotational error requirements per axes of the pointing reference frame.

6.3.2 Pointing error indices

The pointing error e needs to be constrained with respect to its time interval of interest, hence window time or windowed stability, depending on the respective satellite mission application requirements.

6.3.3 Statistical interpretation

The statistical population is chosen in line with the mission application needs according to the three different statistical interpretations of section 5.3.: ensemble, temporal and mixed. Each interpretation requires the description of a different probability characteristic, representing a different random property of the pointing error as discussed in section 5.3.3.

It might be practicable to specify different interpretations with respect to time-random and time-constant sources such that the description represents the required application properties.

NOTE If PES are either time-random or time-constant and not both, the mixed statistical interpretation is equivalent to the ensemble interpretation for time-constant PES and to the temporal interpretation for time-random PES. Thus the mixed statistical interpretation is the suitable approach to express pointing requirements in many satellite missions.

6.3.4 Evaluation period

The evaluation period needs to be specified because different operations might require different pointing error requirements of the same index. Usually an evaluation period corresponds to a satellite operational mode, like "science observation".

6.3.5 Level of confidence

As defined in section 6.2 all requirements have the common mathematical form:

$$\mathrm{Prob}\left[\ |e_{index}| < e_{r,index}\right] \geq P_c$$

where the level of confidence is e.g.:

- 99.7% (general case) or if applicable 3σ (Gaussian distribution)
- 95.5% (general case) or if applicable 2σ (Gaussian distribution)
- 68.3% (general case) or if applicable 1σ (Gaussian distribution).

The use of the 'n x σ' notation for expressing probabilities is restricted to the cases where the pointing error PDF is approximately Gaussian.

For a Gaussian PDF the 95.5% (2σ) bound is twice as far from the mean as the 68.3% (1σ) bound, but this relation does not hold for a general distribution. However, the assumption of Gaussian distribution is valid for many cases in practice due to the central limit theorem, cf. [RD-03], which states that the distribution of the sum of a large number of independent distributed random variables converges to a Gaussian distribution.

NOTE The guidelines of section 4.2.4 in ECSS-E-ST-60-10C need to be regarded when evaluating the level of confidence, because e.g. Gaussian distributed pointing errors per reference frame axis make up a Rayleigh distributed pointing error per LOS.

6.3.6 Power spectral density

In some cases it might not be appropriate to specify a pointing error requirement in terms of its variance but rather with respect to its power spectral density magnitude. This might be necessary because a pointing error requirement specified by its variance σ_R^2 can have several PSD magnitude shapes, as illustrated in Figure 6-1 by the relation:

$$\sigma_R^2 = \int_{f_l}^{f_h} G_R df \quad \text{but also}$$
$$\sigma_R^2 = \int_{f_l}^{f_h} G_R df = \int_{f_l}^{f_h} G_1 df = \int_{f_l}^{f_h} G_2 df = \int_{f_l}^{f_h} G_3 df \tag{6-6}$$

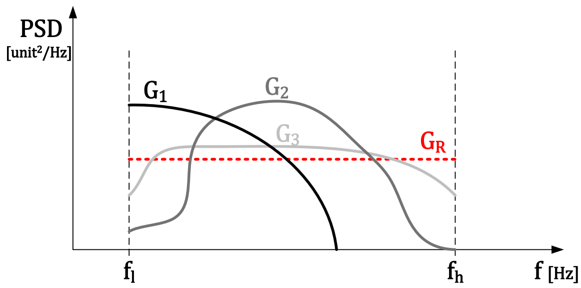

Figure 6-1: PSD Pointing error requirement definition

NOTE In the power spectral density profile requirement, the lower cut-off frequency f_l of the bandwidth usually corresponds to the data acquisition time window (e.g. observation time). The upper cut-off frequency f_h corresponds to the Nyquist frequency, hence sampling time.

6.4 Requirement break-down and allocation

In order to break-down an overall system pointing error to allocate error fractions to different subsystem of the pointing system the usual dilemma is encountered, namely that variances sum quadratic, except when cross-correlated, mean values sum linearly and different PES have different PDF. In the following two approaches for the break-down of the overall system pointing error are introduced:

- **Simple linear break-down:** The overall system pointing error is linearly divided and pointing error fractions are allocated to systems of the pointing system. This has the implication of being conservative because the actual pointing error is assembled of PEC that contribute linear, quadratic or in a correlated manner.
- **Break-down according to probability characteristics:** The overall system pointing error is broken-down according to its probability characteristics with respect to each other. This can be done once the system is analysed based on the pointing error analysis steps AST-1 to AST-3, introduced in section 8 to 10. Hence for break-down the same logic contribution structure applies as for the budgeting.

7 Pointing error analysis methodology

7.1 Approach

An analysis methodology for establishing a pointing error budget is presented in this section 7. The pointing error analysis methodology is adapted to the available PES data and tools as the pointing system design matures. It is a combination of different mathematical elements, analysis methods and mainly two different approaches:

- **frequency-domain approach:** analytic approach restricted to Gaussian processes and linear time-invariant systems to analyse characteristic error properties.
- **time-domain approach:** based on numerical simulations and experimental results to analyse characteristic error properties.

In general, separate pointing error budgets are made for different evaluation periods, e.g. spacecraft operational modes. In this sense also nominal and exceptional budgets should be established. An exceptional budget would for example include specific events like transients that affect the pointing performance in relative short sporadic events. For distinguishing between exceptional and nominal, it is important to take into account the likelihood of occurrence of the exceptional budget.

7.2 Methodology structure

The analysis methodology overall structure with intermediate analysis steps and results is illustrated in Figure 7-1. It is in line with section 7.1, introduced approaches and consists of two different main analysis methods:

a) **simplified statistical method:** analysis with variance, σ, and mean, μ, and their summation per ECSS pointing error index under the assumption of the central limit theorem.

b) **advanced statistical method:** analysis by joint PDF characterization via convolution of different error PDF, $p_{...}(e)$, as described in [RD-04], and evaluation of level of confidence for required ECSS pointing error indices.

In Figure 7-1 method a) is depicted with solid lines whereas the advanced method is depicted as dashed lines. Depending on the available data for the individual steps one or the other method or a combination is applied. The indices letter ‘R’ in Figure 7-1 stands for random variable, the indices letters ‘RP’ stand for random process, the index ‘index’ stands for the ECSS pointing error indices and the index letter ‘B’ stands for bias.

Note that the analysis steps AST-1 to AST-4 are further elaborated and guidelines are introduced in section 8 to 11 of this handbook.

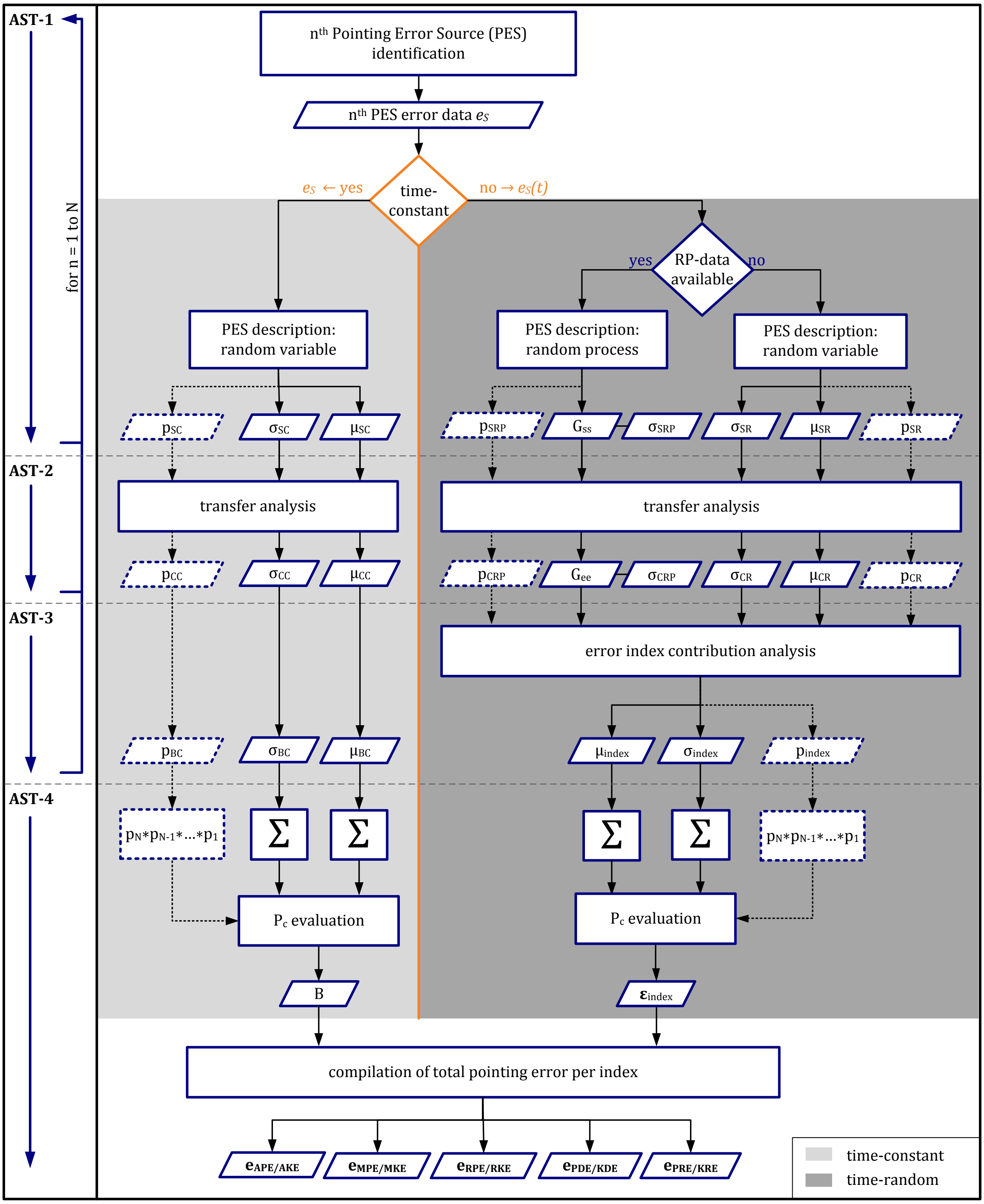

Figure 7-1: Pointing error analysis methodology overview

8 Characterization of pointing error source: AST-1

8.1 Overview

In order to describe and quantify properties for pointing error analysis, PES are analysed according to the methodology in Figure 7-1, and in line with the framework elements in section 5.3. As outlined in Figure 8-1 the characterization of PES, denoted as Analysis Step AST-1, requires the identification of a PES, its categorization in time-constant or time-random and its description as random variable or random process. The handbook only provides guidelines for the description of PES error data because the PES identification is case specific.

NOTE In Annex B examples are provided for different PES error types.

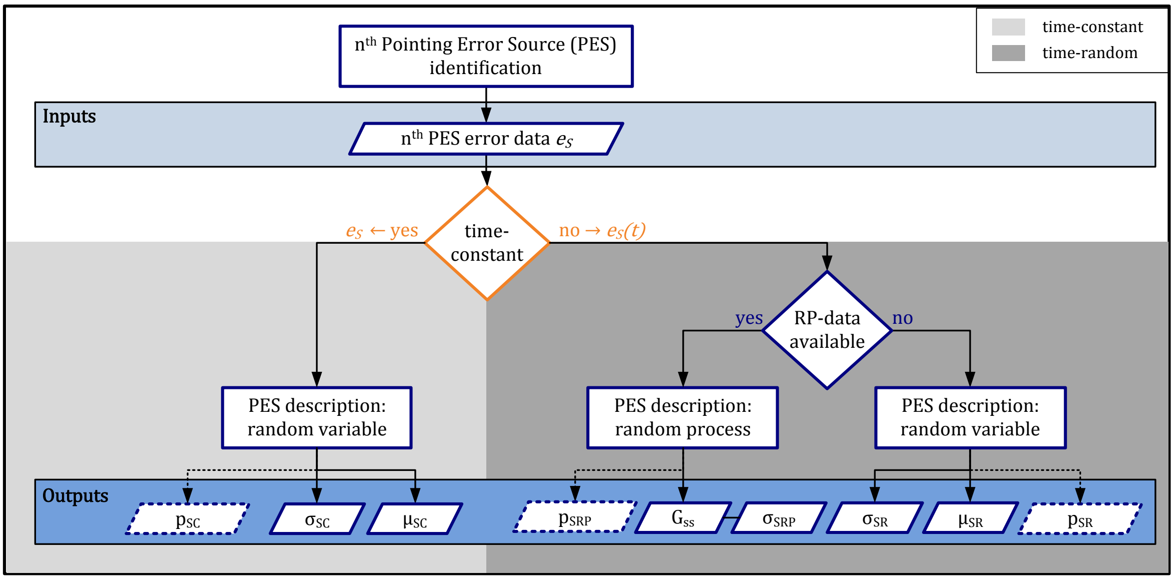

Figure 8-1: Characterization method

8.2 Identification of pointing error source

The identification of characteristic PES error data is essential for a meaningful pointing error analysis. If no detailed PES error data is available, assumptions based on experience need to be made to determine equivalent error data necessary to describe the PES. However, as projects mature hardware for exact error model identification should be made available.

The identification of PES is not subject of this handbook because case specific identification methods needs to be applied, which can be found in literature, cf. [RD-03].

8.3 PES error data classification

PES error data can be classified on the basis of characteristic signal properties as outlined in Figure 8-2. This simplifies the description by the mathematical elements defined in section 5.3.2. If PES error data consists of e.g. non-zero mean Gaussian noise, the data can be separated in two PES, the mean value and zero-mean Gaussian noise error data content. Separating PES error data into signal classes structures and thus simplifies analysis without loss of generality as explained in [RD-03]. This handbook as well as ECSS-E-ST-60-10C provides guidelines for the characterization of bias, periodic, Gaussian random, uniform random and drift error data. Transients and other random error data are usually system specific and thus are not treated in this framework.

NOTE In this handbook a signal is defined as any measurable time-random and/or ensemble-random physical phenomenon.

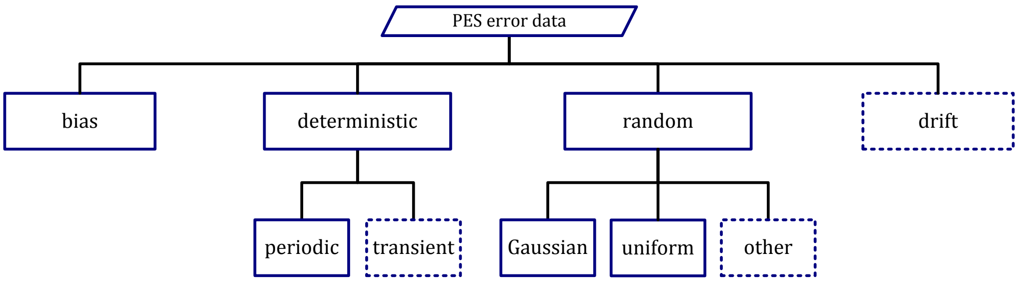

Figure 8-2: PES signal classes

The flow chart in Figure 8-3 provides guidelines for selecting eligible mathematical elements of section 5.3.2 to describe the signal-classified PES error data. As a first step it is distinguished between time-constant and time-random data. This is a natural approach because the different pointing error indices are defined according to time-windowed temporal pointing behaviour, see section 4.2. As shown in Figure 8-3 time-constant PES error data corresponds to a bias that stays constant throughout all operational conditions. Time-random PES error data is either random, periodic or a bias that changes its magnitude e.g. due to different operational conditions.

NOTE A time-constant PES es is constant in time and random with respect to its ensemble of realizations. A time-random PES $es(t)$ is randomly varying in time and can also be random with respect to its ensemble of realizations.

As a second step, it is distinguish whether the time-random PES error data is described by random process theory or by a random variable. This depends on the availability of time series error data and if the random process description criteria in [RD-03] are fulfilled.

The decision tree in Figure 8-3 provides guidelines for systematically selecting a suitable PES description method. In the decision tree, the first criterion categorizes a PES in time-random and time-constant. Time-constant PES do not vary randomly with time, but in their ensemble of realizations. On the other hand, time-random PES have a magnitude that varies randomly in time and/or in its ensemble. A time-constant PES is described as a random variable in line with section 5.3.2.2. A time-random PES is ideally described as a stationary random process if time-series data is available and stationary random process theory is applicable. Therefore guidelines for a stationary random process description are given in section 5.3.2.4. If time series data is not available, ECSS-E-ST-60-10C provides guidelines for an approximate random variable description. Note that describing a PES as stationary random process and thus characterizing its PSD has the advantage that exact time window and stability time properties of the PES, as explained in section 10, are captured. This approach is different compared to ECSS-E-ST-60-10C.

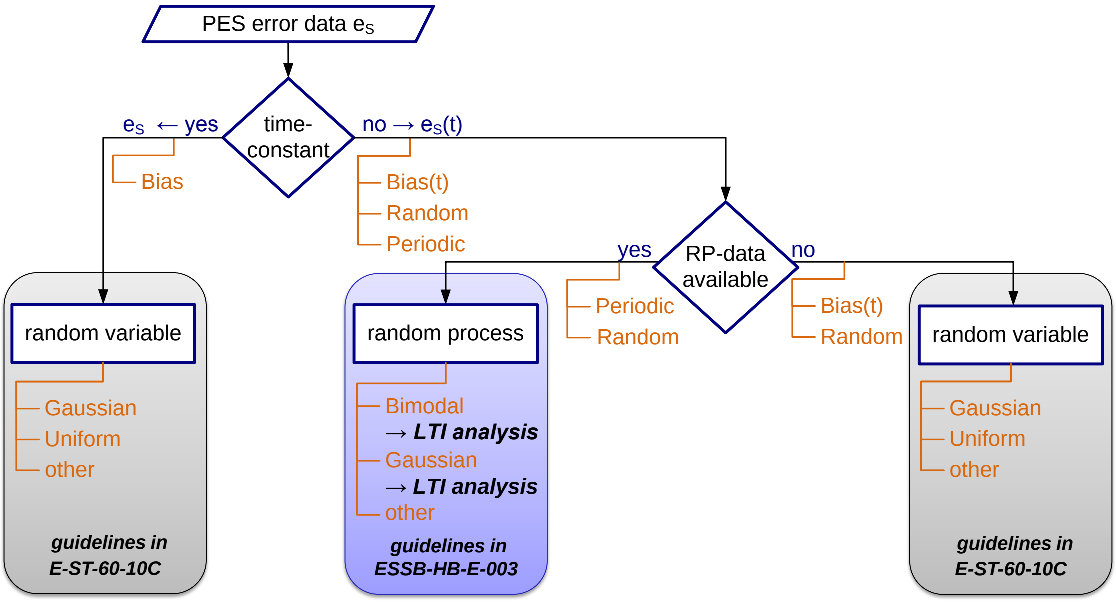

Figure 8-3: PES classification based on error data properties

As can be seen in Figure 8-3, periodic PES error data can also be described with random process theory, cf. [RD-03]. In this respect one might argue that periodic PES error data is deterministic and thus has an exact mathematical formulation. This is true if an exact characterization is possible and the temporal behaviour predictable. In this case it indeed does not contribute to the pointing error because it can be compensated. However, in practice an exact characterization is not always possible and thus clear boundaries between random and deterministic cannot be drawn, i.e. a deterministic error signal might contain variables that take on random values. A sine wave with a random distributed phase is such a signal, cf. [RD-03]. Hence also random uncertainties in deterministic signals contribute to pointing errors.

8.4 Description of PES

8.4.1 Overview

The classification of PES error data is a systematic approach to identify suitable mathematical elements of section 5.3.2 and ECSS-E-ST-60-10C for describing the error data. In the following section, for each case shown in Figure 8-3, description guidelines are provided considering the statistical interpretation defined in the pointing error requirement. Depending on the statistical interpretation, different probability characteristics of the PES error data need to be described.

8.4.2 Time-constant PES description

A bias, which corresponds to a time-constant PES, can be the mean-content of a time-random PES or e.g. an offset due to misalignment. The magnitude of a time-constant PES randomly varies with respect to its ensemble of realizations. This means the bias is stable throughout the mission or varies only once as e.g. the deviations experienced in the LEOP phase (launch vibrations, effect of gravity changes and initial thermal distortions). In any case, during the operational phase of a mission, all these contributors have a constant value for each pointing system (e.g. satellite) realization. They are compensated after in-flight calibration, so only the residual bias is considered for the budgets established after in-flight calibration. Examples of bias-type time-constant PES are given in Table 8-1.

Table 8-1: Time-constant PES examples with signal class

Time-Constant S/C Pointing Error Sources	Signal Class
Misalignment (e.g. payload – star tracker)	Bias
Calibration uncertainty	Bias

The description of the PES error data as bias depends on the statistical interpretation defined with the pointing error requirement. As distinguished in Table 8-2 ensemble and mixed interpretation requires a bias error data description with the underlying probability density function (PDF), $p(e)$. The temporal statistical interpretation considers the maximum value of the bias magnitude, equivalent to the delta distribution $\delta(e_{max})$. For unbounded PDF a maximum value e_{max} is considered such that 99.7% of all magnitude values are smaller than that value. If the PDF is Gaussian, this corresponds to the 3σ value.

In addition to the PDF each time-constant PES signal is described by its statistical properties, mean value and variance.

NOTE An overview of different PDF with their statistical properties is given in Table 5-1 and in more detail in [RD-03].

Table 8-2: Time-constant ensemble-random description

statistical interpret.	random variable description	$p_{SR}(e)$
ensemble	PDF of ensemble-random variable *e(k)*.	$p(e)$
temporal	PDF of ensemble-random variable *e(k)*.	$\delta(e_{max})$ $e_{\max} = \max_k [e(k)]$
mixed	Same as 'ensemble' because temporal behavior is constant.	$p(e)$
ensemble-random variable *e*:	time-constant, but in ensemble randomly varying PES magnitude.	

8.4.3 Time-random PES description

8.4.3.1 General

8.4.3.1.1 Overview

A time-random PES is either random, periodic or a bias that changes due to varying operational conditions. As stated in section 8.3, PES error data can be described as random process if the criteria stated in [RD-03] are fulfilled and time series data is available. Otherwise time-random PES error data is described by a random variable. Examples of time-random PES are given in Table 8-3.

Table 8-3: Time-random PES examples with signal class

Time-Random S/C Pointing Error Sources	Signal Class
Environmental Disturbance Torque Noise	random
Cryogenic Cooler	random
Optical Filter Wheel Motion	transient
Actuator (e.g. Thrusters, Wheel) Noise	random
Drive Mechanisms (solar array, instrument, antenna)	periodic
Reaction Wheel Imbalances	periodic
Attitude Sensor Noise (star tracker, earth sensor, etc.)	bias(t) + random
Inertial Sensor Noise (gyro, accelerometer)	random
Guidance Sensor Noise (radio frequency and optical metrology, GPS)	random
Structure Thermal/Mechanical (due to orbiting)	periodic
Structure Thermal/Mechanical (due to re-orientation)	transient
System dynamics induced errors, e.g.: - sloshing - flexible modes	transient

PES error data can not only be random in time (time-random) but might also have a random ensemble of realization (ensemble-random). This is the case for e.g. a periodic PES with randomly varying amplitude due to changing operational conditions. Each realization of the periodic PES thus has different probability characteristics resulting in an ensemble of realizations. The statistical properties and its PDF are thus random variables themselves.

In addition to the PDF, each time-random PES signal is described by its statistical properties (mean value and variance) and in case of random process description also by its PSD.

8.4.3.1.2 Time-random description in context of statistical interpretation

The description of time-random, but not ensemble-random, PES error data in context of the required statistical interpretation is outlined in Table 8-4. As the error data is time-random only, it can be formulated as $e(k,t)=e(t)$ and $\{e_k(t)\}=\{e(t)\}$.

The ensemble interpretation requires the pointing error to be smaller at all time for a certain percentage of the ensemble of realizations, the maximum error value in time needs to be taken into account. In the temporal interpretation all ensembles of realizations need to be below the pointing error requirement for a certain percentage of time. Due to the fact that the ensemble behaviour per definition is constant, the PDF of the error distribution over time completely describes the error characteristics. The same is valid for the mixed interpretation because it reduces to the temporal interpretation. This is due to the non-existence of a random ensemble variable.

Table 8-4: Time-random description w.r.t. statistical interpretation

statistical interpret.	random variable description	$p_{SR}(e)$	random-process description	$p_{SRP}(e)$
ensemble	Maximum value of time-random variable *e(t)*.	$\delta(e_{max})$ $e_{\max} = \max\limits_{t}[e(t)]$	Maximum value of time-random process *{e(t)}*.	$\delta(e_{max})$ $e_{\max} = \max\limits_{t}[\{e(t)\}]$
temporal	PDF of time-random variable *e(t)*.	$p(e)$	PDF of time-random process *{e(t)}*.	$p(e)$
mixed	Same as 'temporal' because ensemble behavior is constant.	$p(e)$	Same as 'temporal' because ensemble behavior is constant.	$p(e)$
time-random variable *e*: in time randomly varying PES magnitude.				

8.4.3.1.3 Time- and ensemble-random description in context of statistical interpretation

The description of time-random and ensemble-random PES error data in context of the required statistical interpretation is outlined in Table 8-5. As the error data is time-random and ensemble-random, the random variable, describing the PES data depends, on k and t, thus $e(k,t)$. On the other hand, describing the PES data as random process $\{e_k(t)\}$ requires stationary error data.

Ensemble interpretation requires that the pointing error is smaller at all time for a certain percentage of the ensemble of realizations such that the ensemble PDF of the maximum error value e_{max} needs to be taken into account. In the temporal interpretation all ensembles of realizations need to be below the pointing error requirement for a certain percentage of time. This corresponds to the worst case PDF describing the time-random PES magnitude. The mixed interpretation requires a description by conditional probability in terms of random variable description as stated in [RD-03] and ECSS-E-ST-60-10C, or simply by the PDF in terms of stationary random process description.

NOTE If a PES is a non-stationary process that has varying statistical properties due to e.g. different operational conditions, it might be described as stationary process with random statistical properties, i.e. variance $\sigma_e(k)$ and mean $\mu_e(k)$. Hence the statistical properties of the random process would be random themselves and described as variance of the random variance σ_σ and random mean μ_σ with the conditional PDF, $p_k(e)$.

Table 8-5: Time- and ensemble-random description w.r.t. statistical interpretation

statistical interpret.	random variable description		random-process description	
		$p_{SR}(e)$		$p_{SRP}(e)$
ensemble	PDF of random variable *e(k,t)* for worst case error w.r.t. time-random index *t*.	$p(e_{max})$ $e_{\max}(k) = \max_t [e(k,t)]$	PDF of maximum values in time for all realizations k of the stationary random process $\{e_k(t)\}$.	$p(e_{max})$ $e_{\max}(k) = \max_t [\{e_k(t)\}]$
temporal	PDF of random variable *e(k,t)* for worst case error w.r.t. ensemble-random index *k*.	$p(e_{max})$ $e_{\max}(t) = \max_k [e(k,t)]$	PDF of worst case ensemble realization k of the stationary random process $\{e_k(t)\}$.	$p(e_{max})$ $e_{\max}(t) = \max_k [\{e_k(t)\}]$
mixed	PDF of the random variable *e(k,t)*.	$p_k(e) =$ $\int p(e/k)p(k)de$	PDF of stationary random process $\{e_k(t)\}$.	$p(e)$
time-random variable *e*:	in time randomly varying PES magnitude.			
ensemble-random variable with index *k*:	in ensemble randomly varying PDF of time-random variable *e*.			

8.4.3.2 Random process PES description

8.4.3.2.1 Overview

The description of time-random PES error data by random process theory requires characterization of the PDF and PSD. The PDF describes the likelihood of error magnitude occurrences throughout time, but not at which time constants magnitude values occur. That means a PDF does not contain information for analysing and quantifying time-windowed pointing errors. Characterizing the error source also by its PSD provides information about time constants as the PSD describes power magnitude per frequency range. Thus, the PSD of an error source is the basis for time-windowed pointing error analysis as is shown in section 10.

The PES error data is decomposed in mean value and zero-mean process to simplify analysis without loss of generality. Depending on the temporal behaviour the mean values are treated as time-constant bias or time-random bias.

8.4.3.2.2 Periodic

Periodic PES error data is a composition of sine functions, where each sine function stands for a single PES and is described by its frequency f_0, amplitude A_k and phase θ. If the phase θ of a single periodic PES is not or cannot be predicted, it can be assumed to have a uniform PDF between $\pm\pi$. The amplitude A can be a random variable itself depending on the ensemble index k.

The periodic PES can thus be described as pseudo random process:

$$\{e_k(t)\} = \{A_k \sin(2\pi f_0 t + \theta\} \tag{8-1}$$

The random process is stationary (and ergodic) if the amplitude is considered to be equal for each k, meaning that $A_k = A$. In this case the PDF is bimodal:

$$p_k(e) = BM(A_k) = \left(\frac{1}{\pi\sqrt{A_k^2 - e^2}} \right) \quad |e| < A_k\,, \quad \text{otherwise } 0 \tag{8-2}$$

and $p_k(e) = p(e)$. If the amplitude A_k is random, $p_k(e)$ is also random with respect to the ensemble index k. In this case the statistical properties, i.e. mean and standard deviation, are also random variables, $\mu=\mu(k)$ and $\sigma=\sigma(k)$. For $A_k=A$ Table 8-4 applies to determine a correct description of the PES in line with the statistical interpretation defined in the pointing error requirement. Otherwise, meaning that the amplitude varies, Table 8-5 applies.

The power spectral density (PSD) of periodic PES error data is described by:

$$G_{ee}(f) = \frac{A^2}{2}\delta(f - f_0) \tag{8-3}$$

where $\delta(f-f_0)$ is the Dirac-Delta function at the frequency f_0 of the periodic PES.

8.4.3.2.3 Gaussian Random

Gaussian random PES error data is usually noise, which behaviour by definition is not predictable and thus can only be described by random process theory, namely its probability characteristics and power spectral density. There is no cross-correlation between successive realizations and it cannot be decomposed in elementary single periodic errors. Those errors usually originate from actuation noise or sensor measurements as can be seen in Table 4-1.

An ergodic stationary Gaussian random process $\{e(t)\}$ with zero-mean value is completely described by its variance:

$$\begin{aligned}
\sigma_e^2 = E[e^2(t)] &= \int_{-\infty}^{\infty} e^2 p(e)de \quad with \quad p(e) = (\sigma_e\sqrt{2\pi})^{-1}\exp[-\frac{e^2}{2\sigma_e^2}] \\
&= \frac{1}{T}\int_0^T e^2(t)dt \quad for \quad T \to \infty \\
&= \int_{-\infty}^{\infty} G_{ee}(f)df
\end{aligned} \tag{8-4}$$

It should be noted that the PSD of a PES can have any shape, meaning that the noise can be time-correlated.

> NOTE If the PSD G_{ee} of a PES is not available, but only its variance σ_e^2, a first approximation would be to assume the PSD of band-limited white noise with a magnitude of $G_{ee} = \sigma_e^2/f_N$ with f_N being the Nyquist frequency of e.g. sensor noise.

8.4.3.3 Random variable PES description

8.4.3.3.1 Bias(t)

A bias can be the mean-content of a time-random PES or any other offset like the misalignment of structural parts. The time-random bias is time-constant at the timescale of an observation period, but its value can vary randomly at the scale of the pointing system (e.g. satellite) lifetime. This is due to a change in operational conditions, e.g. external influences like temperature and attitude, and not due to time-windowed temporal signal properties. Thus it is possible to have a bias which is constant during one observation, but changes between two different observations. For example, an attitude dependent star sensor error is constant as long as the sensor sees the same area of the sky, but varies when its orientation changes.

The PES error data, that represents a time-random bias, is described as random variable by any PDF and their statistical properties, which are summarized in section 5.3.2 and can be found in [RD-03].

NOTE Time-random behaviour of a PES mean value considered during an observation window Δt is not described by the bias signal class. The random variable description of a bias is necessary if the random changes in the mean value occur based on non-stationary behaviour of the time-random PES.

8.4.3.3.2 **Uniform Random**

Uniform time-random PES error data varies over short timescale, with a uniform PDF in a given range with maximum value *C*, i.e. there is zero probability that an error occurs outside this range. Such errors usually originate in quantisation of a signal, e.g. analogue-to-digital converter resolution errors or errors from a bang-bang controller.

The time-random bias PES error data is described as random variable by a uniform distribution:

$$\begin{aligned} p(e) &= C^{-1} \quad 0 \le e \le C \\ p(e) &= 0 \quad \text{otherwise} \end{aligned} \tag{8-5}$$

with the statistical properties, namely variance and mean:

$$\sigma_e^2 = \frac{C^2}{12} \tag{8-6}$$

$$\mu_e = \frac{C}{2} \tag{8-7}$$

8.4.3.3.3 **Transient**

Transient time-random PES error data is from short-duration non-stationary phenomena with a clearly defined beginning and end, cf. [RD-03]. The error data has complex temporal evolution (e.g. transient oscillations due to moving parts such as rotating mirrors) which cannot be classified in any of the other categories, even after subdivision in elementary error sources.

Depending on the application it might eventually be necessary to specify time requirements for transients to reach stationary conditions due to their temporal limited appearance as it is the case for e.g. transients during re-pointing. In this respect one would define transition periods from stationary to stationary behaviour, rather than including transient-type errors in the budget. Hence a mission and error data specific description method needs to be identified for any PES error data determined to have transient behaviour.

NOTE Henceforth no specific guidelines are provided for transient data within the general pointing engineering framework of this handbook. However to avoid excessive contributions to budgets, it may be desirable to produce two separate budgets: one with (nominal budget) and one without transient contributions (exceptional budget). In some cases two separate sets of requirements may be given for nominal and exceptional performance.

8.4.3.3.4 Drift

Time-random PES error data that varies approximately linearly with time is called drift. The origin is usually a mean or random error integrated over time and thus does not truly represent a separate signal class in the mathematical sense. However in terms of pointing error analysis a separate categorization simplifies analysis.

As stated in [RD-04] a drift error has a linear variation with time:

$$\varepsilon_D = D \cdot \Delta t_D \qquad \textbf{(8-8)}$$

where D is the drift rate in [unit/s]. It is assumed that the drift is reset at intervals Δt_D (e.g. after each observation). Depending on the context, the drift rate D may be a single value, a worst case or an ensemble parameter.

In [RD-04] guidelines are provided for describing drift-type time-random errors as random variables.

9 Transfer analysis: AST-2

9.1 Overview

The description of the PES is given with respect to its point of origin. In order to evaluate a pointing error requirement the transfer of a PES from its origin to the point of interest needs to be analysed to determine the pointing error contributor (PEC). In this context transfer analysis refers to reference frame coordinate transformations as described in ECSS-E-ST-10-09C and to system transformations, e.g. AOCS closed-loop sensor noise transformation. This handbook concentrates on system transformations.

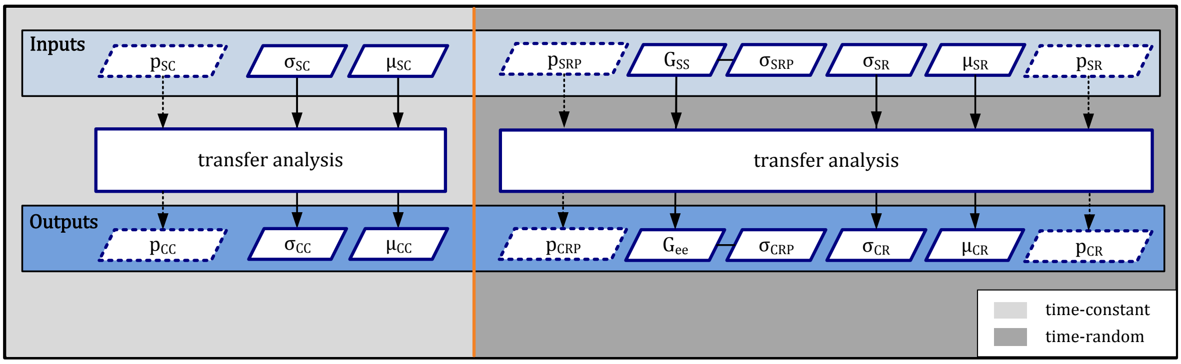

Figure 9-1: Transfer analysis

The input (PES) and output (PEC) parameters of the transfer analysis are shown in Figure 9-1. In the following section an approach for system transfer analysis is introduced that relies on random process theory. Different techniques in frequency and time domain exist for system transformation of time-random PES described as random processes. It can be distinguished between analytic methods, which are based on linear transformation of statistical properties, and numerical methods, which rely on simulations and experimental results.

The system transfer of a PES described as random variables needs to be analysed case by case and cannot be described in a general approach. The payload-star tracker misalignment error would be an example for this case. However, in most cases of random variable description the PES directly corresponds to the PEC and system transfer analysis is not required, except for coordinate transformations.

Time-constant PES system transformation analysis is a simple multiplication of the bias/mean value with the system DC-gain.

NOTE It is recommended to express PEC per axis of a common pointing requirement reference frame in terms of the rotation angle errors: e_x, e_y, e_z or as directional half-cone errors considering the constraints described in section 6.2.

9.2 Frequency-domain

The transfer analysis is concerned with the input-output relation of a system, as comprehensively discussed in [RD-03] and [RD-05]. If the input error signal of a system, the PES, is known and the system can be represented by a linear time-invariant (LTI) transfer function *H(jω)*, being stable and strictly proper, the output error signal, the PEC, can be determined.

According to Eq.(5-13) the variance of a PES described as random processes is related to its PSD by:

$$\sigma^2_{SRP}(e) = \frac{1}{2\pi}\int_0^{\infty} G_{ss}(\omega)d\omega \qquad \textbf{(9-1)}$$

The PSD G_{ss} of the input error signal $e_s(t)$ is transformed by the system according to the relation, cf. [RD-03]:

$$G_{ee}(\omega) = \left|H(j\omega)\right|^2 G_{ss}(\omega) \qquad \textbf{(9-2)}$$

The variance of the output error signal $e_c(t)$ is thus computed from its PSD G_{ee} by:

$$\sigma^2_{CRP} = \frac{1}{2\pi}\int_0^{\infty} G_{ee}(\omega)d\omega \qquad \textbf{(9-3)}$$

The transfer of the PES variance can also be analysed by state-space methods as introduced in [RD-05]. The PES is modelled as the covariance matrix of a Gaussian white noise process with a spectral form filter in state-space representation. With the system transfer matrix the PEC can thus be computed with linear algebra methods.

In order to tune the system transfer function *H(jω)* and to quantify pointing errors, signal and system norms are advantageous. They are introduced in Annex C. In [RD-06] pointing error index specific signal norms are introduced based on [RD-01], [RD-02], and [RD-05] in order to tune closed loop control systems, e.g. the AOCS.

NOTE In terms of analysing the system transfer of a PES, being an ergodic stationary random process with Gaussian distribution, it is only necessary to analyse the PSD transfer because: Gaussian PDF at system input is equal to Gaussian PDF at system output. The same applies if the PES is a periodic pseudo stationary random process.

9.3 Time-domain

In order to analyse the error transfer from the PES to the actual pointing error, time-domain simulations can be run instead of analytic frequency-domain analyses. The described PES PDF (e.g. uniform, Gaussian, bimodal distribution) or PSD in terms of random processes are used to model PES behaviour. Then time-domain simulations are run to get sufficient error sample size, which is statistically representative for characterizing the pointing error contributor PEC at the system output of interest. Each or a joint PEC is then again characterized with respect to its PDF in form of a cumulative histogram and by their PSD.

The topic of simulations is treated in more detail in [RD-04].

NOTE Transfer analysis by simulations does not provide a systematic design framework as the methods used in the analytic analysis. However they are inevitable for system performance verification in terms of Monte-Carlo Simulations.

10 Pointing error index contribution: AST-3

10.1 Overview

Depending on the description of the PES in AST-1 different approaches need to be chosen for the time-windowed pointing error index contribution analysis in line with the pointing error indices defined in section 4.2. The separation of time-random and time-constant PES in AST-1 is actually because of the time-windowed error index contribution analysis described in this section 10. The contribution analysis is shown in Figure 10-1 with input and output parameters.

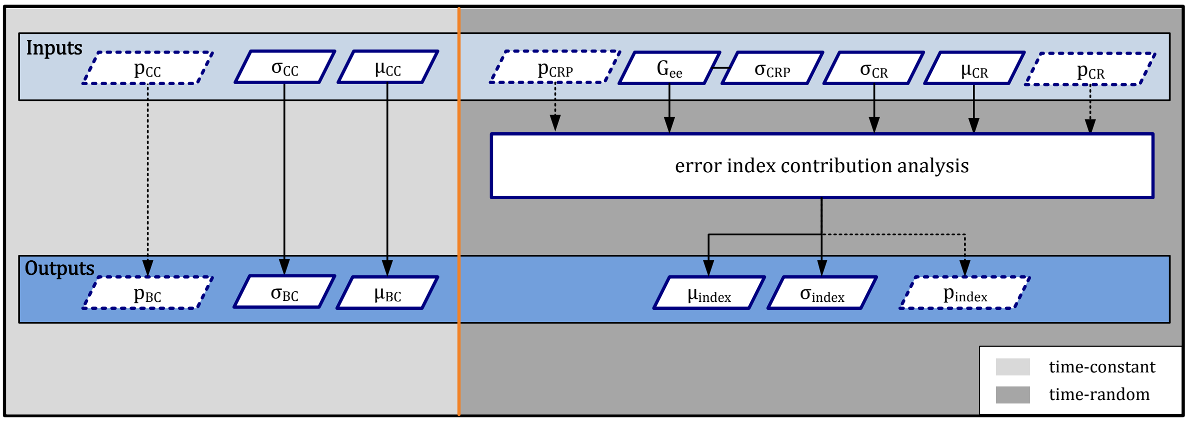

Figure 10-1: Pointing error index contribution analysis

It is not necessary to analyse time-constant PEC because they are by definition independent of time. On the other hand, time-random PEC need to be analysed since their contribution depends on the time window length, Δt and Δt_s.

The random variable description of time-random PES does not include information about time-constants in the error signal because it is only described by its PDF. Nevertheless, in most cases such errors have discrete time changes, meaning that their magnitude change is correlated with a known event. An example would be the attitude-dependent bias of a star tracker. In this case one knows that the bias changes from one observation period with respect to the other, which results in a stability PEC in the PDE or PRE index. Hence, it is important to analyse the statistical properties, quantifying the error index contribution of a PEC described as random variable, case by case. In this respect ECSS-E-ST-60-10C and [RD-04] provide guidelines for different error PDF.

The description of time-random PES as random processes includes information about the temporal behaviour of the PES in form of the PSD. The pointing error index contribution of such a PEC can be quantified by time-windowed PSD metrics, in the following referred to as pointing error metrics.

NOTE In order to analyse a time-random PES with random process theory the PES data needs to be stationary such that its statistic properties are time-invariant. In this case the metric definitions represent a signal norm, cf. [RD-05]. Moreover, in order to completely determine the pointing error PDF, the stationary random process needs to be ergodic as mentioned in section 5.3.2.4.

10.2 Worst case pointing error index

The worst case pointing error indices are defined for the instantaneous pointing errors in Table 4-4. This is necessary if a requirement has different window times and instead of performing analysis for all window times, only the worst case window time is analysed.

Unlike in Table 10-1 in the following it is not distinguished between knowledge error e_K and performance error e_P because for the time-windowed analysis the describing metrics are the same.

Table 10-1: Worst case pointing error index

Pointing Error Indices		
index	***instantaneous***	***worst case***
$e_{APE}(t)$	$= e_P(t)$	*not applicable*
$e_{AKE}(t)$	$= e_K(t)$	*not applicable*
$e_{MPE}(t, \Delta t)$	$= \overline{e_P}(t, \Delta t)$	$= \overline{e_P}(t, \Delta t_{\min})$
$e_{MKE}(t, \Delta t)$	$= \overline{e_K}(t, \Delta t)$	$= \overline{e_K}(t, \Delta t_{\min})$
$e_{RPE}(t, \Delta t)$	$= e_P(t) - \overline{e_P}(t, \Delta t)$	$= \overline{e_P}(t, \Delta t_{\max})$
$e_{RKE}(t, \Delta t)$	$= e_K(t) - \overline{e_K}(t, \Delta t)$	$= \overline{e_K}(t, \Delta t_{\max})$
$e_{PDE}(t, \Delta t_1, \Delta t_2, \Delta t_s)$ $e_{PRE}(t, \Delta t_1, \Delta t_2, \Delta t_s)$	$= \overline{e_P}(t, \Delta t_2) - \overline{e_P}(t + \Delta t_s, \Delta t_3)$	$= \left\langle e_P(t) \right\rangle_{\Delta t_{\min}} - \left\langle e_P(t + \Delta t_{s,\max}) \right\rangle_{\Delta t_{\min}}$
$e_{KDE}(t, \Delta t_1, \Delta t_2, \Delta t_s)$ $e_{KRE}(t, \Delta t_1, \Delta t_2, \Delta t_s)$	$= \overline{e_K}(t, \Delta t_1) - \overline{e_K}(t + \Delta t_s, \Delta t_2)$	$= \left\langle e_K(t) \right\rangle_{\Delta t_{\min}} - \left\langle e_K(t + \Delta t_{s,\max}) \right\rangle_{\Delta t_{\min}}$

Δt_s *stability time where:* $\Delta t_{s,\,max} = max[\Delta t_s]$.

Δt *window time where:* $\Delta t_{min} = min[\Delta t]$, $\Delta t_{max} = max[\Delta t]$.

with $\Delta t = \Delta t_1 = \Delta t_2$

e_{index} *instantaneous error*

$e_K(t)$ *knowledge error signal*

$e_P(t)$ *performance error signal*

time average: $\bar{e}(t, \Delta t) = \left\langle e(t) \right\rangle_{\Delta t} = \dfrac{1}{\Delta t} \displaystyle\int_{t-\Delta t/2}^{t+\Delta t/2} e(t)\ dt$

The PDE and PRE errors cannot be described by time-windowed metrics in a mathematical sense. Time-windowed metrics only exist for a constant window evaluation time Δt as shown in Figure 10-2 and a constant stability evaluation time $\Delta t_s = \Delta t_{PDE/PRE}$ defined by the time difference of two window centres.

However, if there are changing window times for pointing observations, which need to be compared in terms of their stability, one can make an upper bound evaluation according to the worst case MPE window time Δt, corresponding to $min[\Delta t]$, and the worst case stability time $max[\Delta t_s]$. This becomes illustrative considering the time-windowed metric weighting functions in Table 10-3.

NOTE The mathematical definitions of PDE and PRE (seeTable 10-1) are identical. As indicated in Note 5 of section 3.2.10 in ECSS-E-ST-60-10C the difference is in the use of these error indices. The PDE is used to quantify stability among two points in time, which are apart a defined time $\Delta ts = \Delta t_{PDE}$. The PRE expresses the same stability with $\Delta ts = \Delta t_{PRE}$, but among different defined periods, e.g. observation periods.

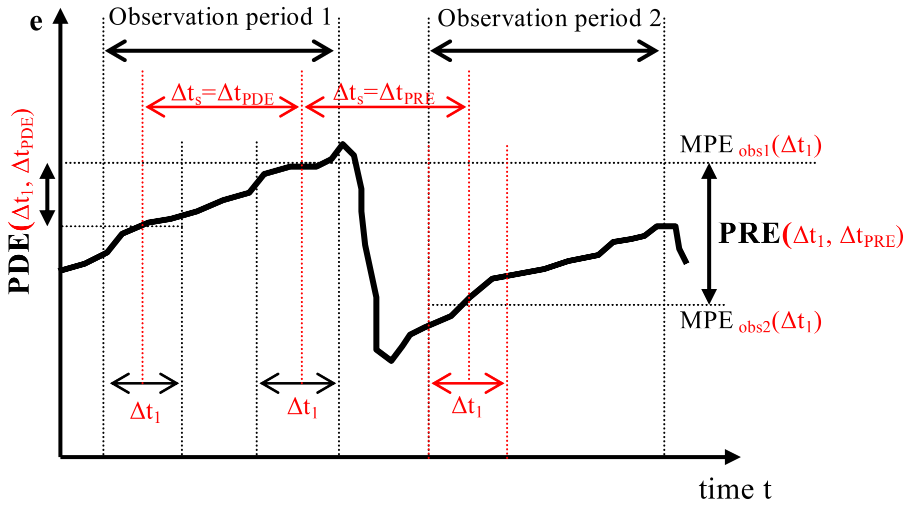

Figure 10-2: PDE/PRE pointing metrics interpretation

10.3 Pointing error metrics

10.3.1 Overview

The time windowed pointing error metrics, derived in [RD-01] and [RD-02], are summarized in Table 10-2. The metrics correspond to different pointing error indices. The absolute, windowed and windowed mean mapping of metrics to indices is straight forward considering the instantaneous worst case error definitions in Table 10-1. Stability and windowed mean stability describe the same set of indices, this is due to the fact that stability is a special case of windowed stability, namely for $\Delta t = 0$. Nonetheless, the stability metrics are listed in Table 10-2 to be complete.

The metrics are computed by evaluating the integrals of the PEC PSD of the zero-mean stationary random process. Note that the zero-mean property of the error is emphasized by including the absolute error signal mean value, μ_{ABS}, in the metrics in Table 10-2.

NOTE 1 In Annex D some notes are provided on the stability and windowed-stability pointing error metric.

NOTE 2 The expected value E[…] in Table 10-2 is computed via the time average if the stationary pointing error process is ergodic. Otherwise it needs to be computed via the ensemble average, cf. [RD-03].

Table 10-2: Pointing error metrics

Pointing Error Metrics		
$\sigma^2_{index} := \sigma^2_{metric}$	*time domain*	*frequency domain*
APE, AKE:= *Absolute (ABS) Metric*	$\sigma^2_{ABS} = E\left[(e(t)-\mu_{ABS})^2\right]$	$= \frac{1}{2\pi}\int_0^\infty G_{ee}(\omega)d\omega$
MPE, MKE:= *Windowed Mean (WM) Metric*	$\sigma^2_{WM}(\Delta t) = E\left[\left(\langle e(t)\rangle_{\Delta t} - \mu_{ABS}\right)^2\right]$	$= \frac{1}{2\pi}\int_0^\infty G_{ee}(\omega)\ F_{WM}(\omega,\Delta t)d\omega$
RPE, RKE:= *Windowed Variance (WV) Metric*	$\sigma^2_{WV}(\Delta t) = E\left[\left\langle \left(e(t)-\langle e(t)\rangle_{\Delta t}\right)^2\right\rangle_{\Delta t}\right]$	$= \frac{1}{2\pi}\int_0^\infty G_{ee}(\omega)\ F_{WV}(\omega,\Delta t)d\omega$
PDE, PRD, KDE, KRE:= *Windowed Mean Stability (WMS) Metric*	$\sigma^2_{WMS}(\Delta t,\Delta t_s) = E\left[\left(\langle e(t)\rangle_{\Delta t} - \langle e(t-\Delta t_s)\rangle_{\Delta t}\right)^2\right]$ $\Delta t=\Delta t_1=\Delta t_2$	$= \frac{1}{2\pi}\int_0^\infty G_{ee}(\omega)\ F_{WMS}(\omega,\Delta t,\Delta t_s)d\omega$
PDE, PRD, KDE, KRE:= *Stability (STA) Metric*	$\sigma^2_{STA}(\Delta t_s) = E\left[(e(t)-e(t-\Delta t_s))^2\right]$	$= \frac{1}{2\pi}\int_0^\infty G_{ee}(\omega)\ F_{STA}(\omega,\Delta t_s)d\omega$

σ^2	*variance*	$e(t)=e_K(t)$	*knowledge error signal*
μ	*mean*	$e(t)=e_P(t)$	*performance error signal*
Δt_s	*stability time*	F_{metric}	*spectral weighting filter*
Δt	*window time*	$G_{ee}(\omega)$	*single-sided PSD [(unit)²/(rad s⁻¹)]*
$E[\ldots]$	*expected value*	$\mu_{ABS} = E[e(t)] = 0$	*PEC mean value*
$\langle e(t)\rangle_{\Delta t} = \frac{1}{\Delta t}\int_{t-\Delta t/2}^{t+\Delta t/2} e(t)dt$	*time average*	$\omega = 2\pi f$	*frequency [rad s⁻¹]*

10.3.2 Time-domain

The following relation is important to understand the nature of the pointing error metrics if mean value and variance of each PEC is considered separately:

$$\psi^2_{ABS} = \mu^2_{ABS} + \sigma^2_{ABS} = \mu^2_{ABS} + \sigma^2_{WM}(\Delta t) + \sigma^2_{WV}(\Delta t) \tag{10-1}$$

Hence a PEC depending on its nature and window time Δt contributes partially to MPE and RPE. An overview of the time windowed pointing error metrics is provided in Table 10-2. This relation is illustrated in the frequency domain in Figure 10-4.

All pointing error metrics are illustrated in the time domain in Figure 10-3.

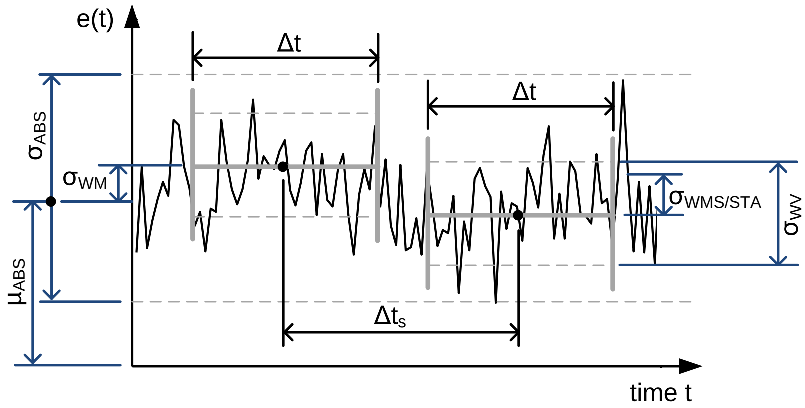

Figure 10-3: Pointing error metrics – Time domain

In Table 10-2 a complete set of time-domain pointing error metrics is listed for time windowed and windowed stability pointing error evaluation as defined in ECSS-E-ST-60-10C.

10.3.3 Frequency-domain

The frequency domain classification of time windowed and windowed stability errors allows an exact evaluation of the metrics defined in section 10.3.2. They can be used to determine the contribution of the PEC signal PSD to the different time windowed and windowed stability errors. Nevertheless, to do so the PSD of the different error sources need to be characterized or with reasonable assumptions approximated.

The frequency-domain pointing error metrics are specific PSD weighting functions F_{metric}. In order to perform analysis, rational approximations, $\tilde{F}_{metric}$, of the weighting functions are given in [RD-02] and summarized in Table 10-3 such that $F_{metric}(\omega) \cong \left|\tilde{F}_{metric}(j\omega)\right|^2$ and with $s = j\omega$. The metrics can be understood as a function by which the PEC signal power, described by its PSD, is weighted. The weighting function corresponds to a low pass, a high pass or a combination of both. As can be seen in Table 10-3 the weighting functions have the form of a sinc-function. This is due to the fact that the windowing in the time domain is equivalent to filtering the time signal by a rectangular function, which has the sinc-function as frequency domain equivalent.

Table 10-3: Pointing error metrics – Frequency domain

<table>
<tr><th colspan="2">Pointing Error Metric Weighting Functions F_{metric}</th></tr>
<tr><td>

Windowed Mean (WM)

$$F_{WM}(\omega,\Delta t)=\frac{2(1-\cos(\omega\Delta t))}{(\omega\Delta t)^2}$$

rational approximation:

$$\tilde{F}_{WM}(s,\Delta t)=\frac{2(s\Delta t+6)}{(s\Delta t)^2+6(s\Delta t)+12}$$

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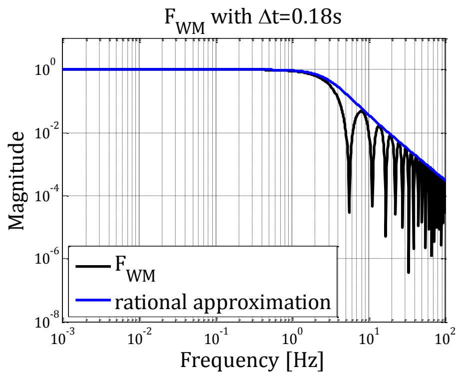

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<tr><td>

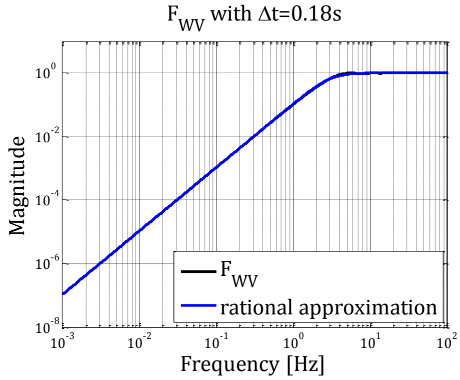
Windowed Variance (WV)

$$F_{WV}(\omega,\Delta t)=1-\frac{2(1-\cos(\omega\Delta t))}{(\omega\Delta t)^2}$$

rational approximation:

$$\tilde{F}_{WV}(s,\Delta t)=\frac{s\Delta t\left(s\Delta t+\sqrt{12}\right)}{(s\Delta t)^2+6(s\Delta t)+12}$$

</td><td>



</td></tr>
<tr><td>

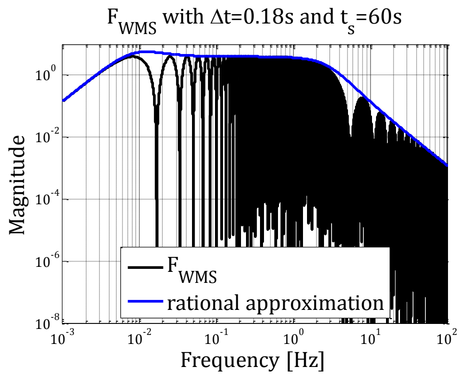
Windowed Mean Stability (WMS)

$$F_{WMS}(\omega,\Delta t,\Delta t_s) = F_{STA}\times F_{WME} = \frac{4(1-\cos(\omega\Delta t_s))^2}{(\omega\Delta t)^2}$$

rational approximation:

$$\tilde{F}_{WMS}(\omega,\Delta t,\Delta t_s)=\tilde{F}_{STA}\times\tilde{F}_{WM}$$

</td><td>



</td></tr>
<tr><td>

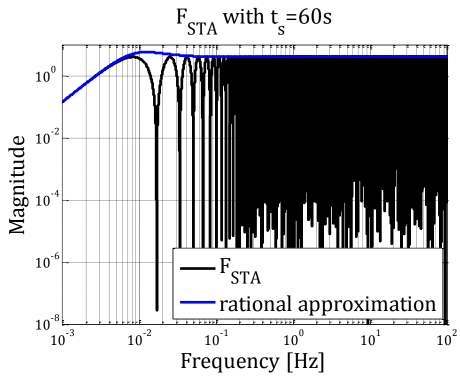
Stability (STA)

$$F_{STA}(\omega,\Delta t_s)=2(1-\cos(\omega\Delta t_s))$$

rational approximation:

$$\tilde{F}_{STA}(s,\Delta t_s)=\frac{2s\Delta t_s(s\Delta t_s+6)}{(s\Delta t_s)^2+6(s\Delta t_s)+12}$$

</td><td>



</td></tr>
</table>

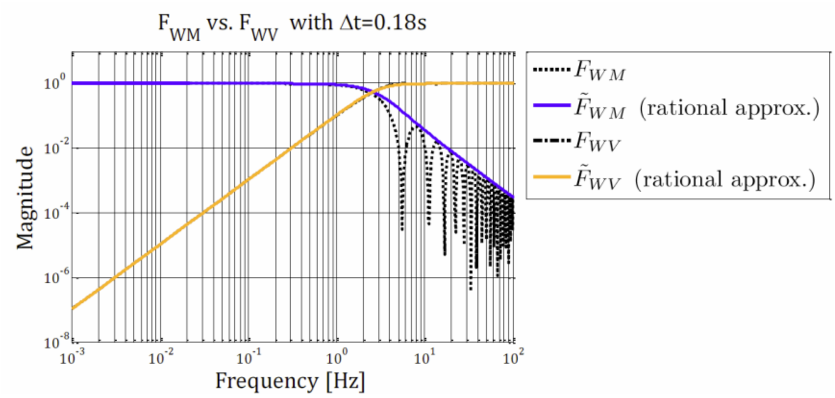


Figure 10-4: Pointing Metric Relation $\sigma^2_{ABS} = \sigma^2_{WM}(\Delta t) + \sigma^2_{WV}(\Delta t)$

10.4 Pointing error index contribution analysis

10.4.1 Time-domain

The analysis of time-windowed pointing error index contribution in simulations can be done by computing the respective expected values, defined in Table 10-2, of the simulation time series. It is practical to implement the rational approximations of the pointing weighting functions as LTI transfer functions in the simulator in order to directly evaluate the pointing error indices during simulations as shown in Figure 10-5.

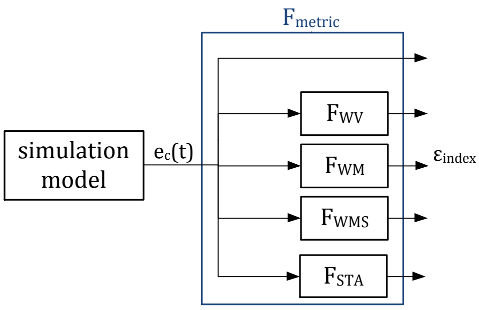


Figure 10-5: Pointing error index evaluation during simulation

A third method to evaluate the pointing errors is to compute the PSD from absolute pointing error time series data and apply the frequency domain PSD weighting filter as in Table 10-2.

10.4.2 Frequency-domain

In ECSS-E-ST-60-10C time classification of the PEC signal is done based on assumptions made upon the frequency of a signal with respect to the window time and windowed stability time. Assumptions of this kind are only possible for e.g. periodic, but not for e.g. Gaussian error signals because in this case the contribution to an error class cannot be estimated without frequency domain information.

The pointing error metrics defined in Table 10-2 provide an accurate, because mathematically justified, method to determine pointing error index contributions by making use of the error PSD, which is characterized in AST-1 and transformed in AST-2.

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The frequency domain integrals of Table 10-2 can be evaluated by numerical integration or state-space methods as shown in [RD-07].

10.5 Statistical interpretation of pointing error indices

During AST-3 it is important to express the statistical properties (i.e. mean and variance) with respect to the statistical interpretation according to the guidelines provided in section 5.3.3 and 8.4. Before continuing with AST-4, this is done for each pointing error index contributor (i.e. μ_{index} or σ_{index}) individually and analogous to the PES statistical interpretation in AST-1.

11 Pointing error evaluation: AST-4

11.1 Evaluation methods

As introduced in section 7.1, there are two methods for analysing pointing performance, the simplified statistical method and the advanced statistical method. In the early development phases, when detailed control design and hardware specifications are not yet available, the performance can be analysed by the simplified statistical method. This method is based on the assumption that the central limit theorem applies.

As the design matures more information of the system (HW and SW) becomes available and the performance can be evaluated by the advanced statistical method, which is more accurate because it considers the actual shape of the PDF. Hence, pointing performance analysis is a combination of the simplified and advanced statistical method changing throughout the development process.

As shown in Figure 11-1 the time-constant and time-random error contributors are first summed separately and the probability with the applicable confidence level is computed. Thereafter, the total pointing error is computed per error index from both intermediate results. These steps are described in this section 11 for the simplified statistical approach. The advanced statistical approach is only briefly introduced.

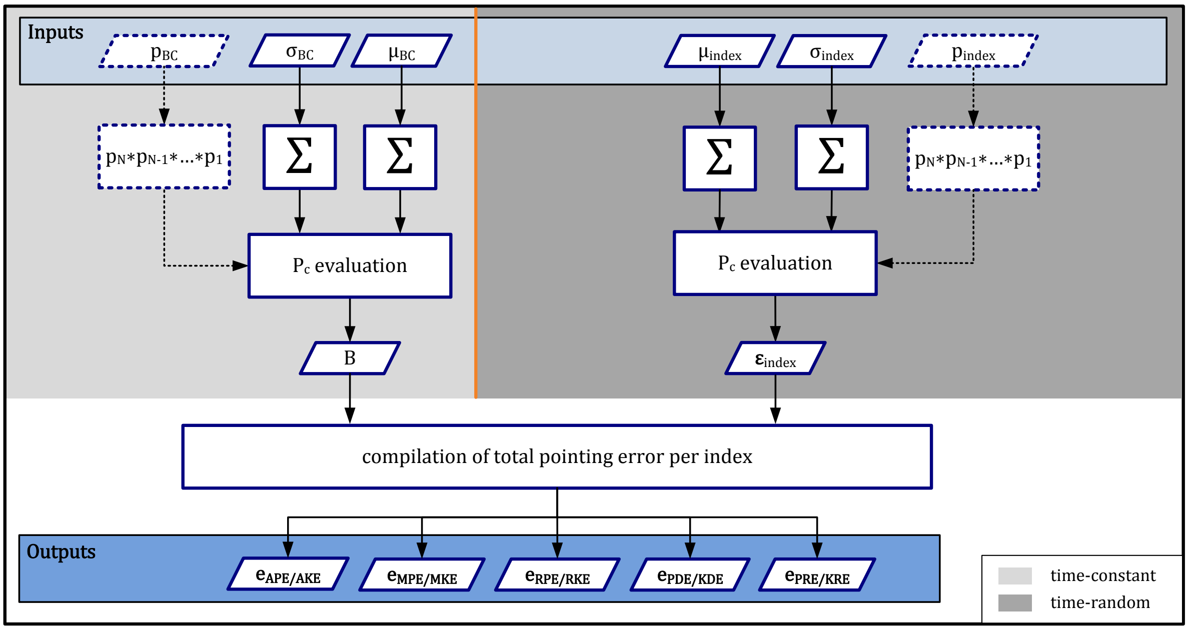

Figure 11-1: Pointing error evaluation

NOTE It is important that the statistical properties (i.e. variance and mean) are expressed in terms of their statistical interpretation before compiling the pointing error budget.

11.2 Simplified method

11.2.1 Time-constant pointing error contributors per index

For time-constant errors, the summation and computation of the applicable probability is identical to the time-random error case, as shown in Table 11-1.

Table 11-1: Time-constant pointing error summation

<table>
<tr><th colspan="4">Time-Constant PEC: Summation and Level of Confidence Evaluation</th></tr>
<tr><th>P_c evaluation</th><th colspan="3">summation</th></tr>
<tr><td rowspan="3">$B \le n_p \times \sigma_B + \lvert \mu_B \rvert$</td><td colspan="3">$\mu_B = \sum_{i=1}^{N} \mu_i$</td></tr>
<tr><td rowspan="2">$\sigma_B^2 \le \sigma_c^2 + \sigma_{uc}^2$</td><td>correlated</td><td>$\sigma_c^2 \le N \sum_{i=1}^{N} \sigma_i^2$ (*1)
or
$\sigma_c^2 \le \left[\sum_{i=1}^{N} \sigma_i \right]^2$ (*2)</td></tr>
<tr><td>uncorrelated</td><td>$\sigma_{uc}^2 = \sum_{i=1}^{N} \sigma_i^2$</td></tr>
<tr><td colspan="4">B bias; P_c level of confidence:
σ^2 variance
μ mean; n_p defines max. error complying with P_c requirement of a Gaussian distribution by $n_p \times \sigma$
N number of summands</td></tr>
<tr><td colspan="4">*1: general worst case upper bound summation (see Annex for derivation).
*2: upper bound summation as in ECSS standard.</td></tr>
</table>

11.2.2 Time-random pointing error contributors per index

In Table 11-2 the summation for the mean and variance of the different time-random pointing error contributions are shown, where:

- the means are summed linearly,
- the uncorrelated variances are summed quadratic,
- for the correlated variances upper bound estimation is used. There are two different computations of the upper bound proposed in Table 11-2, which are derived in Annex E.

Note that the window time, windowed stability time and drift time dependencies are omitted in Table 11-2 to Table 11-1.

Table 11-2: Time-random PEC summation per index

TIME-RANDOM PEC: SUMMATION			
statistics	***summation***		
μ_{index_sum}	$= \sum_{i=1}^{N} \mu_i$		
$\sigma^2_{index\_sum}$	***group summation***	***signal class groups***	
	$\leq \sum_{i=1}^{N} (\sigma^2_{uc})_i + \sum_{i=1}^{N} (\sigma^2_c)_i$	correlated	$\sigma_c^2 \leq N \sum_{i=1}^{N} \sigma_i^2$ *(*1)* *or* $\sigma_c^2 \leq \left[\sum_{i=1}^{N} \sigma_i \right]^2$ *(*2)*
		uncorrelated	$\sigma_{uc}^2 = \sum_{i=1}^{N} \sigma_i^2$
σ^2	*variance*	N	*number of summands*
μ	*mean*		
**1: general worst case upper bound summation (see Annex for derivation).*			
**2: upper bound summation as in ECSS standard → suggested for periodic error summation.*			

The error index is then computed for the applicable confidence level. The method is shown in Table 11-3. First the standard deviation is multiplied by n_p, where n_p is a positive scalar such that for a Gaussian distribution the n_p confidence level encloses the probability P_c, as specified in the requirement. Then the standard deviation is summed with the mean values.

NOTE Although the application of the central limit theorem is assumed, it is recommended to check its applicability. In case of e.g. a dominant periodic error the compilation of the budget should be adapted accordingly or the advanced statistical method applied.

Table 11-3: Level of confidence evaluation for time-windowed PEC per index

Time-Random PEC: Level of Confidence Evaluation		
index	***P_c evaluation assuming applicability of central limit theorem***	***P_C requirement***
APE(ΔtD)/ AKE(ΔtD)	$\varepsilon_{index} = n_p \cdot \sigma_{index,sum}(\Delta t_D) + \lvert\mu_{index}\rvert$	$\mathrm{Prob}(\lvert e\rvert < \varepsilon_{index}) \geq P_c$
MPE(Δt, ΔtD)/ MKE(Δt, ΔtD)	$\varepsilon_{index} = n_p \cdot \sigma_{index,sum}(\Delta t, \Delta t_D) + \lvert\mu_{index}\rvert$	
RPE(Δt, ΔtD)/ RKE(Δt, ΔtD)	$\varepsilon_{index} = n_p \cdot \sigma_{index,sum}(\Delta t, \Delta t_D)$	
PDE(Δt, Δts, ΔtD)/ KDE(Δt, Δts, ΔtD)	$\varepsilon_{index} = n_p \cdot \sigma_{index,sum}(\Delta t, \Delta t_s, \Delta t_D)$	
PRE(Δt, Δts)/ KRE(Δt, Δts)	$\varepsilon_{index} = n_p \cdot \sigma_{index,sum}(\Delta t, \Delta t_s)$	

e	*pointing error*	P_c	*level of confidence*
ε_{index}	*max. time-random error compliant with* P_c	n_p	*defines error that complies with* P_c *requirement of a Gaussian distribution by* $n_p \times \sigma_{index}$
μ_{index}	*mean of time-random error*		
Δt_D	*drift re-set time*		

11.2.3 Compilation of total pointing error per index

Finally, the time-constant and time-random pointing errors are summed per APE, AKE, MPE and MKE indices. On the other hand, the time-constant error does not contribute to the other pointing error indices. An overview of the summation is shown in Table 11-4.

Table 11-4: Compilation of total pointing error per index

Total Pointing Error per Index	
index	***compilation***
APE(Δt_D)/ AKE(Δt_D)	$e_{index} = B + \varepsilon_{index}(\Delta t_D)$
MPE(Δt, Δt_D)/ MKE(Δt, Δt_D)	$e_{index} = B + \varepsilon_{index}(\Delta t, \Delta t_D)$
RPE(Δt, Δt_D)/ RKE(Δt, Δt_D)	$e_{index} = \varepsilon_{index}(\Delta t, \Delta t_D)$
PDE(Δt, Δt_s, Δt_D)/ KDE(Δt, Δt_s, Δt_D)	$e_{index} = \varepsilon_{index}(\Delta t, \Delta t_s, \Delta t_D)$
PRE(Δt, Δt_s)/ KRE(Δt, Δt_s)	$e_{index} = \varepsilon_{index}(\Delta t, \Delta t_s)$
e_{index}	*max. error per index complying with* P_c
Δt_s	*stability time*
Δt	*window time*
Δt_D	*drift re-set time*

11.3 Advanced method

This method corresponds to the so-called exact error combination method as defined in ECSS-E-ST-60-10C. In order to apply the statistical method, each individual source is characterised in terms of probability distribution function (e.g. uniform, Gaussian, bimodal distribution). Then time simulations are performed to get a sufficient sample size representing the statistical behaviour of all error sources and of their potential cross-correlation. This is often designated as “Monte Carlo simulations”. For each run, pointing performance is estimated and when all runs are performed, the results are presented in the form of a cumulative histogram giving the expected performance as a function of the cumulated number of samples. Then, the RMS (1σ) performance corresponds to the value which is not exceeded for 68.3% of the temporal samples. The 2σ performance corresponds to the value which is not exceeded for 95.5% of the temporal samples. The worst case (3σ) performance corresponds to the value which is not exceeded for 99.7% of the temporal samples.

Monte Carlo simulations and the related statistics are further elaborated in [RD-04].

12 Conclusion

The ESA PEE Handbook provides an engineering step-by-step process ranging from the formulation of system pointing error requirements, to systematic pointing error analysis, and eventually to the compilation of pointing error budgets for compliance verification. The process is consistent with *ECSS-E-ST-60-10CC* standard ECSS-E-ST-60-10C and complements it with additional elements like the PSD characterization and the pointing error metrics. Moreover, it defines an interface for the unambiguous formulation of pointing error requirements and provides guidelines, recommendations and examples for specific case of satellite pointing error engineering. As the process has clearly separated steps with defined input and output data it is quite generic and thus applicable to any mission type and design phase.

The ESA PEE Handbook together with the ECSS standard replaces the "old" ESA Pointing Error Handbook [RD-08]. Furthermore, it is intended to complete the ESA PEE Handbook in the future by one or several documents providing guidelines for the mapping process, i.e. the flow-down of application requirements (e.g. from the ESA MRD) to pointing error requirements.

Annex A Pointing scene

Considering a pointing scene, as shown in Figure A-1, the pointing error is described with respect to the target plane. Translational pointing errors on the target plane are a result of rotational errors about the pointing system axes. Hence rotational errors about pointing system x- and y-axis correspond to a displacement error $e_{x\theta}$ and $e_{y\Phi}$ on the target plane respectively. Rotational errors about the z-axis for any point $\mathbf{p}$ on the target plane, however, produce rotational errors, that eventually can be mapped into displacement errors along e_x and e_y or vice versa by:

$$\mathbf{e} = \begin{pmatrix} e_x \\ e_y \end{pmatrix} = \mathbf{R}_{\Psi}(\mathbf{p} + \mathbf{e_l}) - \mathbf{p} \qquad \text{wih} \qquad \mathbf{p} = \begin{pmatrix} x_p \\ y_p \end{pmatrix} \text{ and } \mathbf{R}_{\Psi} = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix} \qquad \text{[A-1]}$$

where the directional boresight, line of sight (LOS) or half-cone error $\mathbf{e_l}$ is defined as:

$$\mathbf{e_l} = \begin{pmatrix} e_{x\theta} \\ e_{y\phi} \end{pmatrix} \quad \text{with} \quad |\mathbf{e_l}| = \sqrt{e_{x\theta}^2 + e_{y\phi}^2} \qquad \text{[A-2]}$$

Hence the resulting pointing error is described by two translations and one rotation one the target plane or by three rotations about the body axes of the pointing system.

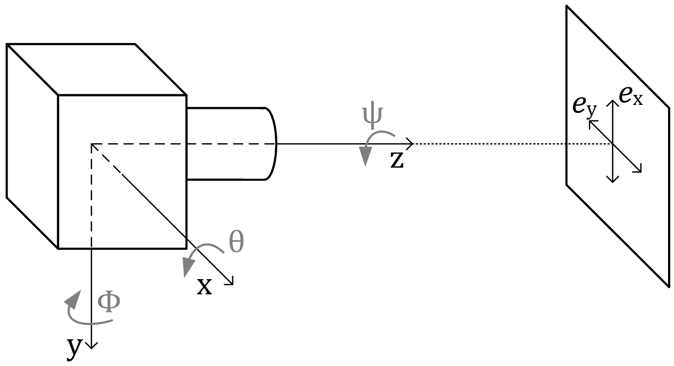

Figure A-1: Pointing scene

x, *y*, *z*: pointing system axes with respective rotation angles θ, Φ, ψ.

e_x, e_y: LOS error coordinates on target plane.

NOTE Rotational errors with a Gaussian distribution about the x- and y-axis result in a Rayleigh distribution (see Table 5-1) about the LOS, cf. ECSS-E-ST-60-10C.

Annex B
Pointing error description using different statistical interpretations

B.1 Satellite pointing example

In this Annex B a simplified satellite pointing example is illustrated to support the introduced methodology in this handbook. The satellite system is shown schematically in Figure B-1.

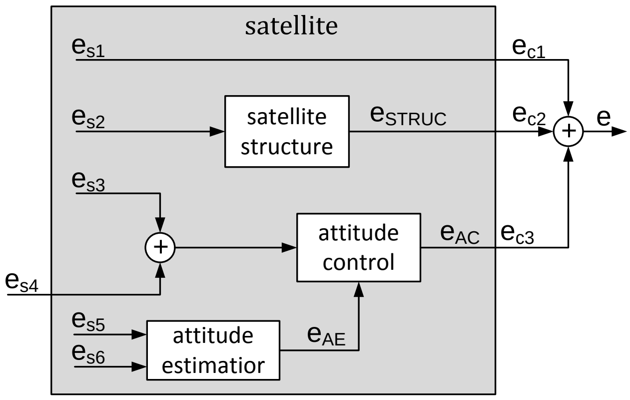

Figure B-1: Satellite pointing example

The satellite system is affected by the PES listed in Table B-1.

Table B-1: PES in satellite pointing example

PES	Type
e_{s1}	Payload-star tracker misalignments
e_{s2}	µVibrations (outside of control bandwidth)
e_{s3}	Reaction wheel errors
e_{s4}	External disturbances
e_{s5}	Star tracker errors
e_{s6}	Gyro errors

In Table B-2 three exemplary PES are taken from the satellite pointing example in Figure B-1 and Table B-1. They are described in line with the analysis method introduced in section 7. Two different misalignment errors are taken into account to show the difference between an ensemble-random PES and the more complex case of a time- and ensemble-random PES. For the AOCS gyro-stellar estimator noise two different cases are treated, in one the PES is time-random only and in the other one the PES is time-random and ensemble-random.

In AST-1 to AST-3 the pointing errors (PES, PEC and error index contributor) can be categorized as in Table B-3. Note that the attitude estimator PEC, which is a PES for the attitude control, is the transformation of the gyro and star tracker PES by the attitude estimator. Further details on Star Tracker PES are provided in section 5.5 of ECSS-E-ST-60-20C Rev 1.

Table B-2: PES examples

Pointing Error Sources or Contributors – PES or PEC	
Payload-STR misalignment	Time-constant misalignment errors (STR internal, Payload internal, structure) with uniformly distributed error values among an ensemble of satellites (before in-orbit calibration)
Payload-STR thermo-elastic distortion	Payload-STR thermo-elastic misalignment errors with sinusoidal (assumed) variation of different periods (cross-correlated or none cross-correlated). In addition, the amplitudes have uniformly distributed uncertainty errors due to seasonal effects.
AOCS gyro-stellar estimator bias	Attitude knowledge bias error originating from star tracker pixel and FOV spatial errors. In a first approach, the bias error can be assigned a uniform distribution because different stellar configurations are assumed to have equal probability.
AOCS gyro-stellar estimator noise	Attitude knowledge Gaussian noise originating from gyro and star tracker temporal noise. In a first approach the RMS noise power is time-constant.
AOCS gyro-stellar estimator noise -attitude dependent	Attitude knowledge Gaussian noise originating from sensor measurement noise. The RMS noise power has a uniformly distributed uncertainty error due to different operational conditions.

Table B-3: Categorization of pointing errors examples

PES/PEC	temporal behavior	signal class	time-random variable	description	ensemble-random variable
Payload-STR misalignment	time-constant	bias	none	random variable	misalignment bias
Payload-STR thermo-elastic distortion	time-random	periodic	misalignment magnitude	zero-mean stationary random process with bi-modal PDF	uniformly distributed amplitude A of periodic variation due to different operational conditions
AOCS gyro-stellar estimator bias	time-random	bias(t)	bias magnitude	random variable	none
AOCS gyro-stellar estimator noise	time-random	random	noise error magnitude	zero-mean stationary random process with Gaussian PDF	none
AOCS gyro-stellar estimator noise-attitude dependent	time-random	random	noise error magnitude	zero-mean stationary random process with Gaussian PDF	uniformly distributed noise RMS value σ due to different operational conditions

B.2 Time-constant description

The time constant pointing error (PES and PEC) are described in Table B-4 according to their categorization and signal class by means of the respective statistical properties and their PDF.

Table B-4: Time-constant pointing error description

time-constant PES/PEC	statistical interpret.	random variable description			
			$p_{SC}(e)$	$\mu_{SC}(e)$	$\sigma_{SC}(e)$
Payload-STR misalignment	ensemble	Sigma value of uniformly distributed misalignment error.	$U(0, e_{max})$	0	σ_U
	temporal	Maximum misalignment error.	$\delta(e_{max})$	C	0
	mixed	Same as 'ensemble' because temporal behavior is constant.	$U(0, e_{max})$	0	σ_U

The mixed (or ensemble) interpretation allows a realistic statistical approach in the quadratic summation of constant biases in the last AST-4 step, while the temporal approach forces an arithmetic summation of all worst case elementary biases.

B.3 Time-random pointing error description by a random variable

The time-random pointing error (PES and PEC) are described in Table B-5 according to their categorization and signal class by means of the respective statistical properties and their PDF.

Table B-5: Time-random pointing error description by random variable theory

time-random PES/PEC	statistical interpret.	random variable description			
			$p_{SR}(e)$	μ_{SR}	σ_{SR}
AOCS gyro-stellar estimator bias	ensemble	Worst case bias error.	$\delta(e_{max})$	e_{max}	0
	temporal	Sigma value of uniformly distributed bias error.	$U(0, e_{max})$	0	σ_U
	mixed	Same as 'temporal' because ensemble behavior is constant.	$U(0, e_{max})$	0	σ_U

B.4 Time-random pointing error description by a random process

The time-random pointing error (PES and PEC) are described in Table B-6 according to their categorization and signal class by means of the respective statistical properties and their PDF.

Table B-6: Time-random pointing error description by random process theory

time-random PES/PEC	statistical interpret.	random process description			
			$p_{SRP}(e)$	μ_{SRP}	σ_{SRP}
AOCS gyro-stellar estimator noise	ensemble	Maximum noise magnitude (Gaussian noise is unbounded => 3 sigma value).	$\delta(e_{max})$ $e_{max}=3\sigma_G$	e_{max}	0
	temporal	Sigma (or zero-mean RMS) value of Gaussian noise process.	$G(0,\sigma_G)$	0	σ_G
	mixed	Same as 'temporal' because ensemble behavior is constant.	$G(0,\sigma_G)$	0	σ_G

B.5 Time-random and ensemble-random pointing error description by a random process

The time-random pointing error (PES and PEC) are described in Table B-7 according to their categorization and signal class by means of the respective statistical properties and their PDF.

Table B-7: Time- and ensemble-random PES description by random process theory

time-random PES/PEC	statistical interpret.	random process description		
			$p_{SRP}(e)$	σ_{SRP}
Payload-STR thermo-elastic distortion	ensemble	Sigma value of uniformly distributed misalignment variation amplitude.	$U(0, e_{max})$ $e_{max} = A_{max}$	σ_U
	temporal	Sigma (or zero-mean RMS) value of periodic misalignment variation with maximum amplitude.	$BM(e_{max})$ $e_{max} = A_{max}$	σ_{BM}
	mixed	Sigma value of uniformly distributed sigma (or zero-mean RMS) noise value of uniformly distributed periodic misalignment variation with amplitude A.	$\int p(e\|A)p(A)de =$ $\int BM(A)U(0, A_{max})de$	$\sigma(A)$
AOCS gyro-stellar estimator noise - attitude dependent	ensemble	Sigma value of uniformly distributed maximum noise magnitude (Gaussian noise is unbounded => 3 sigma value).	$U(0, e_{max})$ $e_{max} = 3\sigma_{G,\,max}$	σ_U
	temporal	Sigma (or zero-mean RMS) value of Gaussian noise process with maximum sigma.	$G(0, \sigma_{G,max})$	σ_G
	mixed	Sigma value of Gaussian ensemble-random and time-random noise process if stationary.	$G(0, \sigma_G)$	σ_G

Annex C Signal and system norms for pointing error analysis

C.1 Overview

Signal norms are convenient for measuring the size of a pointing error signal entering and leaving a system. On the other hand system norms are convenient to determine the gain for such a system transformation of pointing signal norms. Considering e.g. the AOCS of a satellite, PES entering and PEC leaving the system can be quantified in terms of signal norms to determine the AOCS closed loop pointing error performance. In this sense several norms exist to measure the size of a pointing error signal. The selection of the appropriate norm depends on the nature of the signal and its requirement.

It is important to mention that norms considered in this document have *seminorm* properties as defined in [RD-05].

C.2 Signal norms

In the following some important signal norms are introduced:

L_1-norm:

$$\|e\|_1 := \int_0^\infty |e(t)| dt \qquad \text{[C-1]}$$

The **L_1**-norm defines the absolute integral error of a signal. It is useful to measure the total amount of consumables in a system, e.g. propellant consumption of a satellite.

L_2-norm:

$$\|e\|_2 := \left(\int_0^\infty e(t)^2 dt \right)^{1/2} \qquad \text{[C-2]}$$

The **L_2**-norm is a measure of total signal energy. It is appropriate to e.g. measure the size of a transient signal occurring during satellite re-orientation.

L_p-norm:

$$\|e\|_p := \left(\int_0^\infty |e(t)|^p dt \right)^{1/p} \quad with\, 1 \le p < \infty \qquad \text{[C-3]}$$

The **L_p**-norm is the general formulation of the **L_1**- and **L_2**-norm.

L_∞-norm:

$$\|e\|_\infty := \sup_{t\geq 0}|e(t)| \quad \text{[C-4]}$$

The $\mathbf{L_\infty}$-norm corresponds to the absolute peak value of a signal. It can be used to limit the absolute value of a signal, e.g. a satellite tracking error.

RMS-norm:

$$\|e\|_{rms} := \left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e(t)^2\,dt\right)^{1/2} \quad \text{[C-5]}$$

The RMS-norm squared is a measure of the steady-state power of a signal. Note that even though a signal has a small RMS-norm it might be large for some short time periods.

RMS-norm for stationary random processes:

time-domain: $\|e\|_{rms} := \left(E[e(t)^2]\right)^{1/2}$

frequency-domain: $\|e\|_{rms} := \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{ee}(\omega)\,d\omega\right)^{1/2}$ [C-6]

with E[…] being the expected value and $S_{ee}(\omega)$ the double-sided PSD in [unit^2/rad s^{-1}]. The general definition of the RMS-norm in Eq. [C-5] is equal to the RMS-norm of stationary random processes in Eq. [C-6] if the stationary random process is ergodic. The RMS-norm is a suitable measure of e.g. the expected PEC if the PES is a random stationary noise process.

C.3 System norms

12.1.1.1.1 Induced System Norms

The induced system norm corresponds to the maximum gain of a system and is specified as:

$$\|H(j\omega)\|_{p,ind} := \sup_{\|e_s\|_p\neq 0}\frac{\|e_c\|_p}{\|e_s\|_p} = \sup_{\|e_s\|_p\neq 0}\frac{\|He_s\|_p}{\|e_s\|_p} \quad \text{[C-7]}$$

The induced system norm measures the size of the output PEC, $e_c(t)$, for the worst case input PES, $e_s(t)$. Thus it allows analysing the system transformation of pointing error signals as required in section 9. This is possible if the pointing error signal is quantified with the above signal norms and the system has a linear, time-invariant, causal, and finite-dimensional transfer function, $H(j\omega)$.

In the following some important induced system norms are introduced:

L_2-norm induced:

$$\|H(j\omega)\|_{2,ind} = \|H(j\omega)\|_\infty := \sup_{\|e_s\|_p\neq 0}\frac{\|e_c\|_2}{\|e_s\|_2} = \sup_{\|e_s\|_p\neq 0}\frac{\|He_s\|_2}{\|e_s\|_2}$$

$$with\ \ \|H(j\omega)\|_\infty := \sup_{\omega\neq 0}|H(j\omega)| \quad \text{[C-8]}$$

The induced $\mathbf{L_2}$-norm corresponds to the $\mathbf{H_\infty}$-norm and represents the energy-gain for the PES system transfer.

L_∞-norm induced:

$$\|H(j\omega)\|_{\infty,ind} = \|h(t)\|_1 := \sup_{\|e_s\|_\infty \neq 0} \frac{\|e_c\|_\infty}{\|e_s\|_\infty} = \sup_{\|e_s\|_\infty \neq 0} \frac{\|He_s\|_\infty}{\|e_s\|_\infty}$$
$$with\ \|h(t)\|_1 = \int_0^\infty |h(t)|dt \qquad [C-9]$$

The induced **L_∞**-norm corresponds to the **L_1**-norm of the impulse response, *h(t)=δ(t)*, of the transfer function *H(jω)*. It represents the system gain for the PES signal peak value transformation.

RMS-norm induced:

$$\|H(j\omega)\|_{rms,ind} = \|H(j\omega)\|_\infty := \sup_{\|e_s\|_{rms} \neq 0} \frac{\|e_c\|_{rms}}{\|e_s\|_{rms}} = \sup_{\|e_s\|_{rms} \neq 0} \frac{\|He_s\|_{rms}}{\|e_s\|_{rms}}$$
$$with\ \|H(j\omega)\|_\infty := \sup_{\omega \neq 0} |H(j\omega)| \qquad [C-10]$$

The induced RMS-norm corresponds to the **H_∞**-norm and represents the RMS-gain for the PES system transfer.

H_2-norm:

In case the PES is a random stationary noise process with a certain PSD $S_{ee}(\omega)$ the RMS-norm of the output PEC is:

$$\|e_c\|_{rms} = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} |H(j\omega)|^2 S_{ee}(\omega) d\omega \right)^{1/2} \qquad [C-11]$$

If $S_{ee}(\omega)=1$, the RMS-norm of $e_c(t)$ corresponds to system **H_2**-norm:

$$\|H(j\omega)\|_2 := \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} |H(j\omega)|^2 d\omega \right)^{1/2} \qquad [C-12]$$

The **H_2**-norm is a measure of system gain with white noise input. It is not an induced system norm, but it is relevant in many practical cases where a PES can be approximated as white noise. Furthermore, by Parseval's theorem the **H_2**-norm of a system corresponds to the **L_2**-norm of the system's impulse response signal, *h(t)*:

$$\|e_c\|_2 = \left(\int_0^\infty |h(t)|^2 dt \right)^{1/2} = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} |H(j\omega)|^2 d\omega \right)^{1/2} = \|e_c\|_{rms} \quad , with\, e_c(t) = h(t). \qquad [C-13]$$

C.4 Summary

The system norms in Eq. [C-8] to [C-13] are defined in [RD-05] and [RD-09]. A summary of system gains for relating certain PES signals to PEC signal norms is given in [RD-09].

Table C-1: System gains for certain input signals and output signal norms

		input $e_s(t)$	
		$e_s(t)=\delta(t)$	$e_s(t)=\sin(\omega t)$
output $e_c(t)$	$\lVert e_c(t)\rVert_2$	$\lVert H\rVert_2$	∞
	$\lVert e_c(t)\rVert_\infty$	$\lVert H\rVert_\infty$	$\lvert H\rvert$
	$\lVert e_c(t)\rVert_{rms}$	0	$\frac{1}{\sqrt{2}}\lvert H\rvert$

If the input PES signals are described by signal norms, [RD-09] provides the respective system norms for PES to PEC input-output signal norm relations.

Table C-2: System gains for different input-output signal norms

		input $e_s(t)$		
		$\lVert e_s(t)\rVert_2$	$\lVert e_s(t)\rVert_\infty$	$\lVert e_s(t)\rVert_{rms}$
output $e_c(t)$	$\lVert e_c(t)\rVert_2$	$\lVert H\rVert_\infty$	∞	∞
	$\lVert e_c(t)\rVert_\infty$	$\lVert H\rVert_2$	$\lVert h\rVert_1$	∞
	$\lVert e_c(t)\rVert_{rms}$	0	$\leq\lVert H\rVert_\infty$	$\lVert H\rVert_\infty$

NOTE The RMS-norm corresponds to the standard deviation σ of a signal with zero mean value. Hence transfer analysis in the frequency domain as introduced in section 9.2 implicitly uses signal and system norms.

Annex D
Notes on pointing error metrics

D.1 Windowed mean stability (WMS) metric

The windowed mean stability (WMS) metric for determining the pointing error index PDE, PRE, KDE, KRE is a combination of the windowed mean (WM) metric and the stability (STA) metric. This becomes obvious by looking at the PSD in Figure D-1 of the pointing error weighted by the metric weighting filter, F_{STA} and F_{WME}.

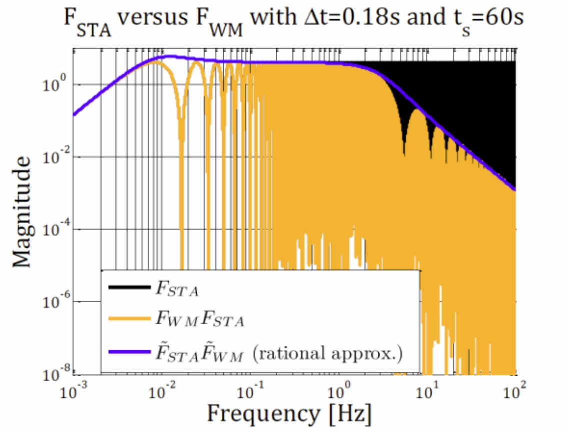

Figure D-1: Pointing error metric weighting function relation: $F_{WMS} = F_{WM} \times F_{STA}$

D.2 Relation Allan Variance and WMS metric (PDE/PRE)

The Allan Variance, commonly used to describe performance errors of gyros and frequency stability errors in clocks, is a specific case of the windowed stability metric described in Table 10-2. Considering the PSD weighting filter of Table 10-3 and setting $\Delta t_s = \Delta t$ we get:

$$\begin{aligned} F_{WMS}(\omega\Delta t) &= F_{STA}(\omega\Delta t)\cdot F_{WM}(\omega\Delta t) \\ &= 2(1-\cos(\omega\Delta t))\frac{2(1-\cos(\omega\Delta t))}{(\omega\Delta t)^2} \\ &= \frac{\sin^4(\omega\Delta t/2)}{(\omega\Delta t/2)^2} \end{aligned} \qquad \text{[D-1]}$$

Taking this weighting function and inserting it in the WMS weighting function results in the Allan Variance:

$$\sigma_{WMS}^{2}(\Delta t) = \frac{1}{2\pi}\int_{0}^{\infty} 2 \cdot S_{ee} \frac{\sin^{4}(\omega\Delta t/2)}{(\omega\Delta t/2)^{2}} d\omega \qquad \text{[D-2]}$$

D.3 Transformation from Allan Variance to PSD

This transformation is possible, but with the consequence of major loss of accuracy. This is because the process of Allan Variance needs to be split into sections of common inclination. It can be decided between different inclinations (noise types), that have different mathematical formulations to transform the Allan Variance to a PSD. The approach can be found in detail in [RD-010].

Annex E Notes on summation rules

E.1 Overview

Analyzing pointing error contributors also requires the analysis of their cross-correlation when they are effective at the same time. However, a cross-correlation analysis of pointing error contributors might not always be possible. In the following rules are derived for the summation of pointing error contributors. The errors are considered to be zero-mean errors (e_A, e_B, e_C), that can either be described as random process or random variable.

E.2 Sum of mean square values for cross-correlated errors

In this section E.2 a formula is derived for summing errors that have an unknown degree of cross-correlation. This is done by considering the worst case cross-correlation among errors such that an upper bound summation is performed.

The variance of the sum of errors is described by the expected value:

$$\begin{aligned}\sigma_e^2 = \; E[e^2(t)] = \; & E[e_A^2(t)] && + E[e_A(t)e_B(t)] && + E[e_A(t)e_C(t)] \\ & + E[e_B(t)e_A(t)] && + E[e_B^2(t)] && + E[e_B(t)e_C(t)] \\ & + E[e_C(t)e_A(t)] && + E[e_C(t)e_B(t)] && + E[e_C^2(t)] \\ \textit{with} \quad & e(t) = e_A(t) + e_B(t) + e_C(t) \end{aligned} \qquad \text{[E-1]}$$

If the degree of cross-correlation is unknown, the following inequality can be applied:

$$\begin{aligned} & E[e_x(t)] \pm 2E[e_x(t)e_y(t)] + E[e_y(t)] \geq 0 \\ & \left|E[e_x(t)e_y(t)]\right| \leq \frac{1}{2}\left(E[e_x(t)] + E[e_y(t)]\right) \end{aligned} \qquad \text{[E-2]}$$

thus it follows that:

$$\sigma_e^2 = E[e^2(t)] \; = 3E[e_A^2(t)] + 3E[e_B^2(t)] + 3E[e_C^2(t)] = 3\sigma_{e_A}^2 + 3\sigma_{e_B}^2 + 3\sigma_{e_C}^2 \qquad \text{[E-3]}$$

In the general case that means:

$$\begin{aligned}\sigma_e^2 = E[e^2(t)] \; & = N \cdot E[e_1^2(t)] + N \cdot E[e_2^2(t)] + \ldots + N \cdot E[e_N^2(t)] \\ & = N\sigma_{e_1}^2 + N\sigma_{e_2}^2 + \ldots + N\sigma_{e_N}^2 \\ & = \mathrm{N}\sum_{\mathrm{i}=1}^{\mathrm{N}} \sigma_i^2 \end{aligned} \qquad \text{[E-4]}$$

E.3 Sum of mean square values for none cross-correlated errors

In this section E.3 a formula is derived for summing errors that have no cross-correlation.

If the errors are not correlated, then the following relation applies:

$$\begin{aligned} \sigma_e^2 &= E[e^2(t)] = E[e_A^2(t)] + E[e_B^2(t)] + E[e_C^2(t)] \\ with \quad & e(t) = e_A(t) + e_B(t) + e_C(t) \end{aligned} \qquad \text{[E-5]}$$

meaning that the cross-correlation terms vanish because they are zero. Thus the general summation rule is:

$$\begin{aligned} \sigma_e^2 = E[e^2(t)] &= E[e_1^2(t)] + E[e_2^2(t)] + \ldots + E[e_N^2(t)] \\ &= \sigma_{e_1}^2 + \sigma_{e_2}^2 + \ldots + \sigma_{e_N}^2 \\ &= \sum_{i=1}^{N} \sigma_i^2 \end{aligned} \qquad \text{[E-6]}$$